Forcing and colorings in a given Baire class

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Question

Can we generalize this theorem to any countable ordinal?

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2. $\Delta(X) \cap \overline{R}^{\tau_\xi \times \tau_\xi} \neq \emptyset$

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• This gives, for every limit $\lambda \leq \xi$ and uniformly, a recursive increasing and cofinal sequence (λ_n) in λ

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• Let $\mathcal{L}_\xi$ be the collection of leaves of $\mathcal{T}_\xi$ (the nodes of rank 0)

- Let $x \in 2^{\mathcal{L}_\xi}$

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- We define a digraph $\mathbb{J}_\xi$ on $\mathbb{X}_\xi$ as follows:

$$\mathbb{J}_\xi := \big\{ \big((x, s0\gamma), (x, s1\gamma) \big) \in \mathbb{X}_\xi^2 \mid |s| = n^x < \omega \big\}$$

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3. *there is a homomorphism h from $(\mathbb{X}_\xi, \mathbb{J}_\xi)$ into (X, R) such that $h^{-1}(O) \in \mathbf{\Sigma}^0_\xi$ for every τ_ξ-open set O*

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We set, for $p \in \mathbb{P}$, $[p] := \{x \in 2^{\mathcal{L}_\xi} \mid p \subset L^x\}$

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   - if φ is a Borel code, then $\neg\varphi$ is a Borel code
2. If φ is a Borel code, then $[\varphi]$ is the Borel subset of $2^{\mathcal{L}_\xi}$ coded by φ:
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allowing us to prove the weak dichotomy

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Let P *be a* $\boldsymbol{\Pi}^0_3$ *subset of* $[\subseteq]$. *Then* P *is strongly represented.*

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