Fiid coloring random maps

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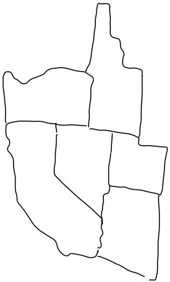
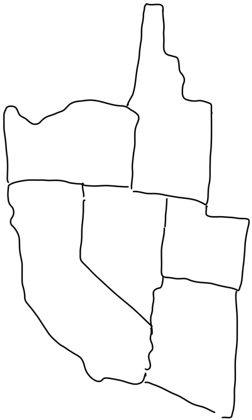
(*joint with* Justin Hsu, Daniel Sium)

CalTech Logic Seminar, May 2026

I. Local color

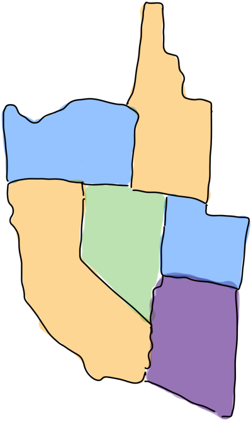


Recall the 4-color theorem:



How many colors do you need to color a map so that neighboring regions get different colors?

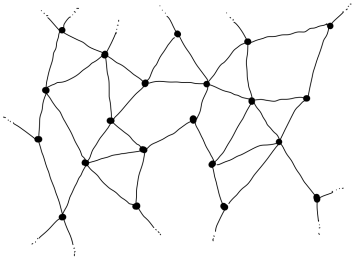
Recall the 4-color theorem:



Theorem (Appel–Haken)

Every planar graph can be 4-colored.

What if our graph is only locally planar?



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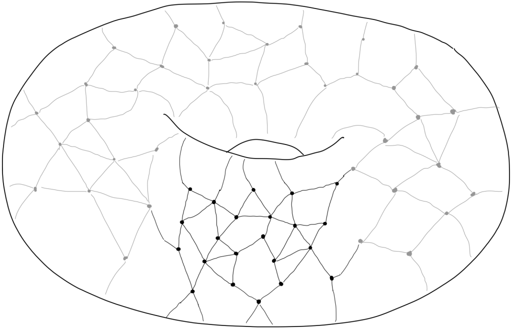
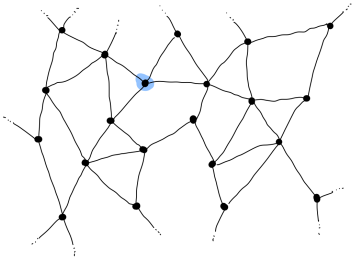


What if we want a local coloring procedure?



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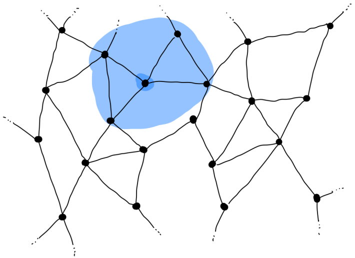

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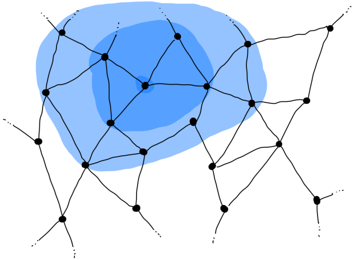

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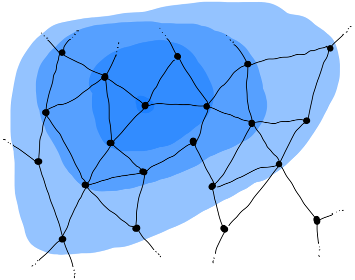
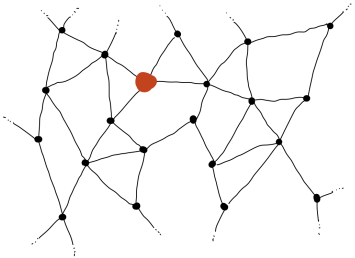


What if we want a local coloring procedure?



By work of Elek, Bernsteyn, and other, these questions are also related to problems about coloring graphs measurably, continuously, with Borel colorings, in $L(\mathbb{R})$, etc.

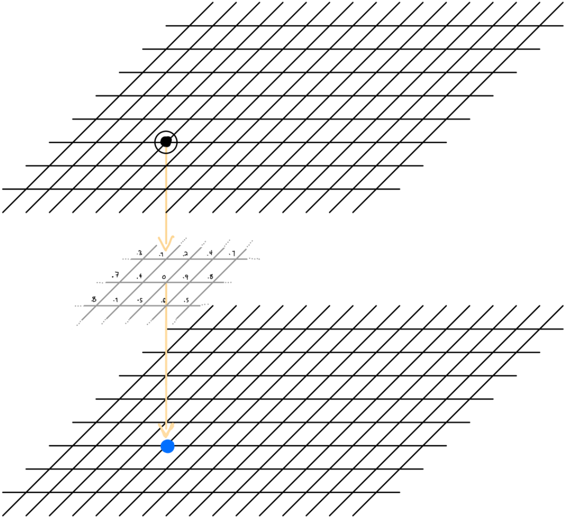
By work of Hatami, Lovasz, Szegedy, and others, these questions are also related to questions about sparse graph limits.

The answers to local questions are determined by the *geometry* of the graph, not just the topology.

We can use fiid processes to model local procedures:

Definition

A Γ-fiid random k-labelling of $G = \text{Cay}(\Gamma)$ is a measurable Γ-equivariant map from the Bernoulli shift $([0,1]^\Gamma, \lambda^\Gamma)$ to k^Γ.



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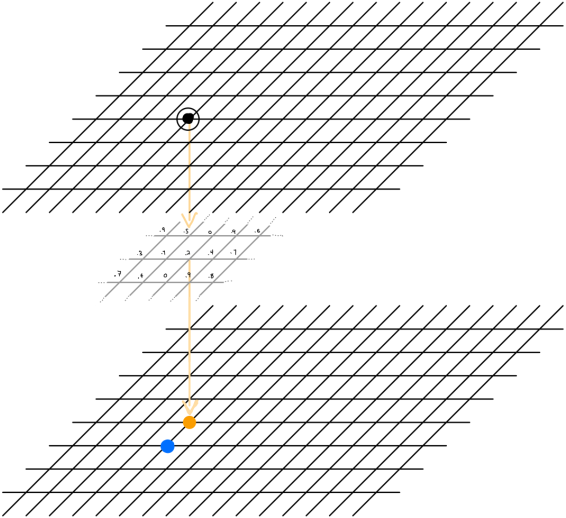

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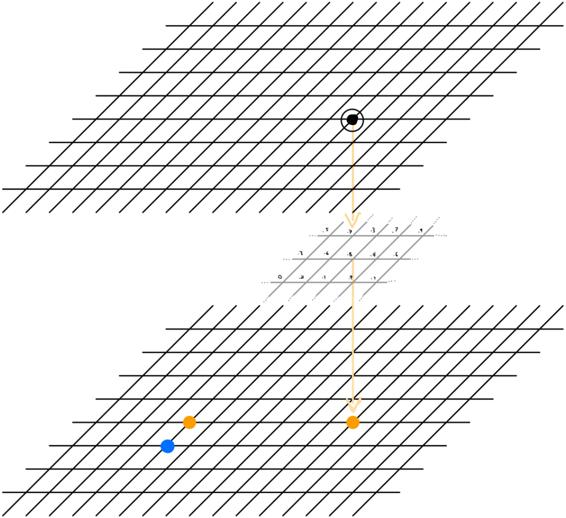
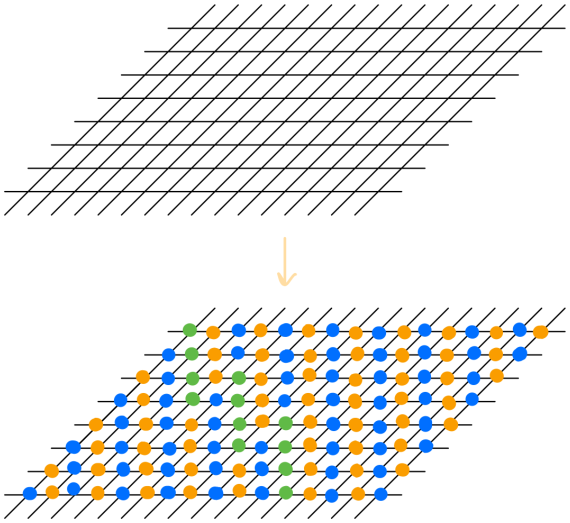
.3 .1 .2 .4 .7
.7 .4 0 .9 .8





Definition

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Note: A Γ-fiid labelling of G is equivalent to a measurable labelling of the Γ-Bernoulli shift.

II. Percolation

Definition

For a graph G, G_p is p-bond percolation on G.

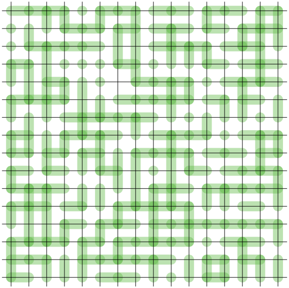


Figure: $(\mathbb{Z}^2)_{.5}$

For any (countable connected) G, G_p undergoes phase shifts:

- For $p < p_c$, almost surely every component of G_p is finite
- For $p_c < p < p_u$, almost surely there are infinitely many infinite components in G_p
- For $p_u < p$, almost surely there is exactly one infinite component in G_p.

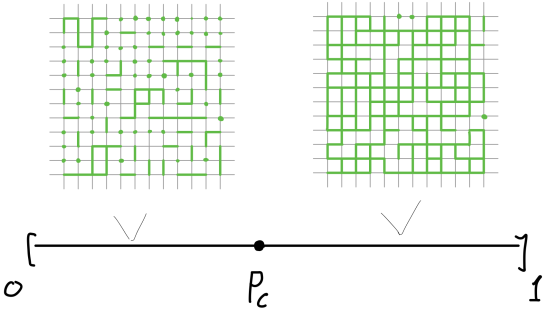


Figure: For $\mathbb{Z}^2$ $p_c = p_u = 1/2$

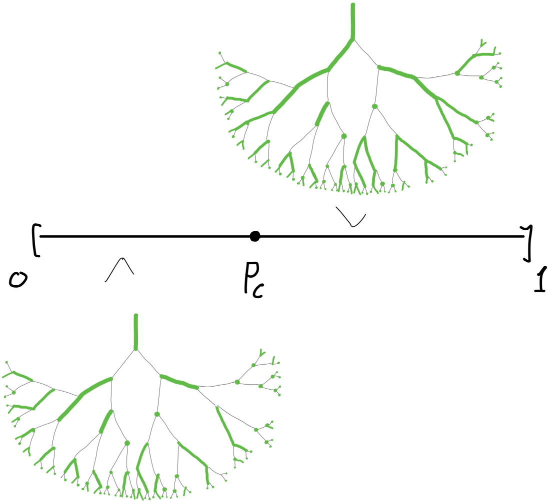


Figure: For a regular tree, $p_c < p_u = 1$

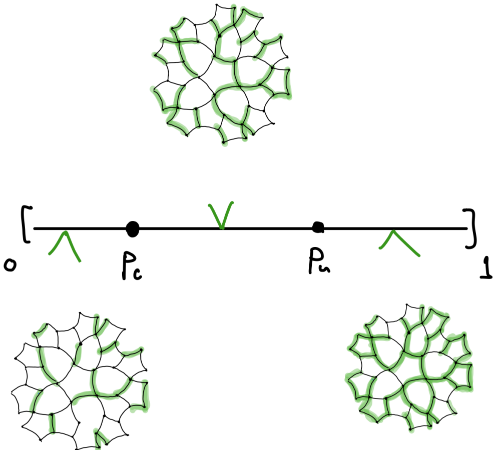
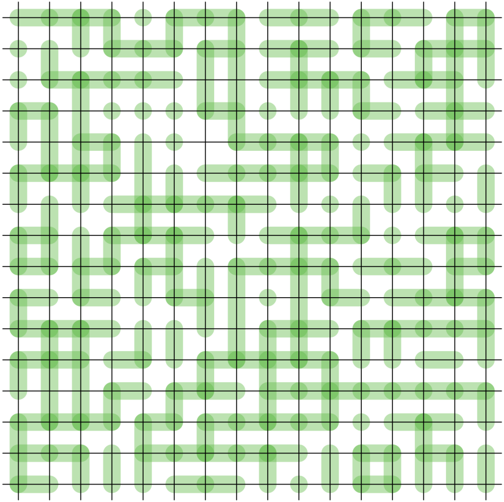


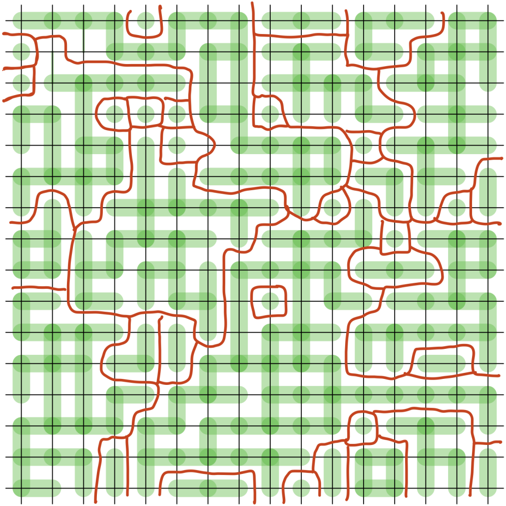
Figure: There are graphs with $0 < p_c < p_u < 1$

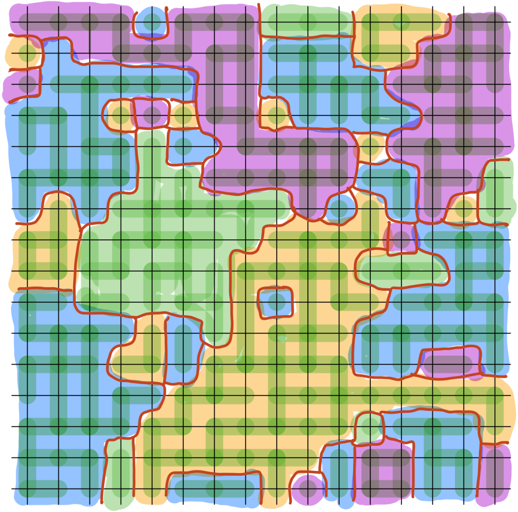
Conjecture (Benjamini–Schramm)

For any transitive nonamenable G, $p_c < p_u$.

III. Our problem







Definition

For a graph G, E_G is the G-connectedness relation. Define the percolation quotient $\tilde{G}_p = G/E_{G_p}$ (throwing away self-loops and parallel edges).

For $G = \mathrm{Cay}(\Gamma)$, how many colors do we need to Γ-fiid color $\tilde{G}_p$?

Definition

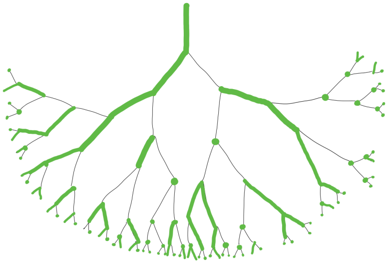
$\chi_\mu(\tilde{G}_p)$ is the least k so that there is an fiid k-labelling of G which is E_{G_p}-invariant and assigns different labels to G-neighbors in different G_p components.

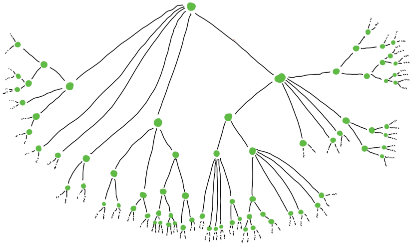
Note: for $p < p_c$, this is equivalent to $\chi_\mu(\mathcal{G})$ for some pmp $\mathcal{G}$ (almost the same holds for $p > p_u$).

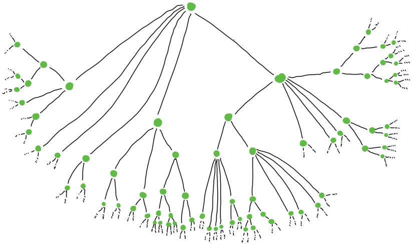
Proposition

For $p_c < p < p_u$, $\tilde{G}_p$ does not admit a fiid coloring.

IV. Trees







Proposition

If $p < p_c(T_n) = 1/(n-1)$ *then* $(\widetilde{T_n})_p$ *is a unimodular Galton-Watson tree after reweighting appropriately.*

Proposition

Suppose G_i, G are unimodular Galton–Watson trees.

1. *If $\mathbb{E}(\deg_G(v)) < \infty$, then G has a finite fiid-coloring.*
2. *If $\mathbb{E}(\deg_{G_i}(v)) \to \infty$, then $\chi_\mu(G_i) \to \infty$.*

(These are not quite converses as the bound in (1) is not uniform in $\mathbb{E}(\deg_G(v))$.)

So, what about our $(\widetilde{T_n})'_p$s? (Or indeed, non-amenable graphs in general).

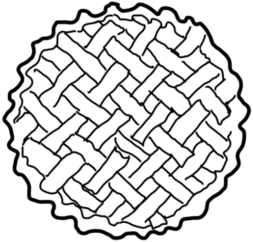
Theorem

For any unimodular transitive planar G,

$$\sup_{p \le p_c, p > p_u} \chi_\mu(\tilde{G}_p) < \infty.$$

In fact, for any nonamenable unimodular planar G and $p > p_u$, $\chi_\mu(\tilde{G}_p) = 4$.

V. Grids



Let's consider $G = \mathbb{Z}^2$. (Or $\mathrm{Cay}(\mathbb{Z}^2, \{\vec{e}_0, \vec{e}_1\})$ or to be pedantic).

Theorem

For all p, $\chi_\mu((\widetilde{\mathbb{Z}^2})_p) \leq 5$.

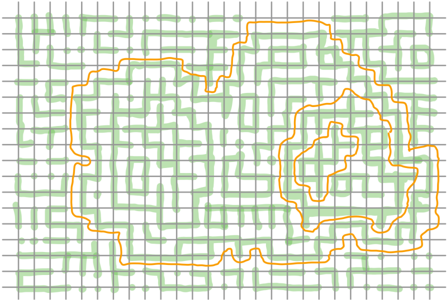
Note: hyperfiniteness and the 4-color theorem give an upper bound of 7.

The proof is in 2 cases: $p \geq 1/2 = p_c = p_u$ and $p < 1/2$.

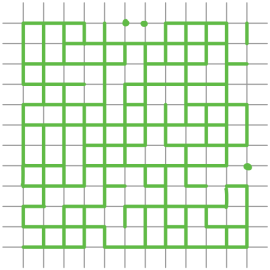
Case 1: $p \geq 1/2$.

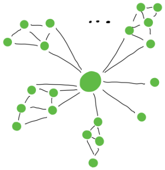
Theorem (Russo–Seymour–Welsh)

For $p \geq p_c$, every component of G_p is completely surrounded by a larger component.

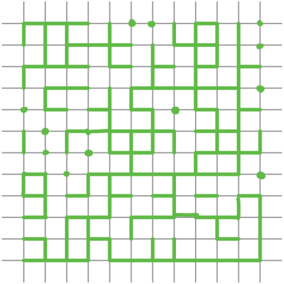


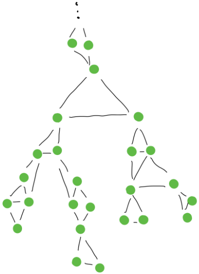
$p > 1/2$





$p = 1/2$





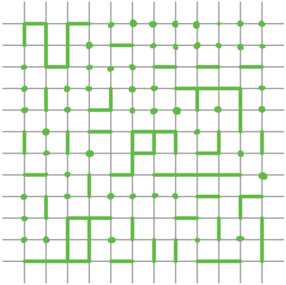
Case 2: $p < 1/2$.

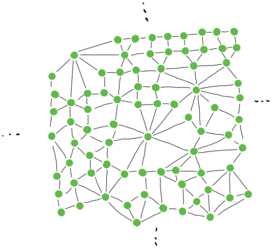
Theorem (Aizenmann–Barsky)

For $p < 1/2$, the probability of seeing a component of size r decays exponentially in r, i.e.

$$\mathbb{P}\big(\mathrm{diam}([x]_{G_p}) > r\big) < O(c^r)$$

for some $c < 1$.



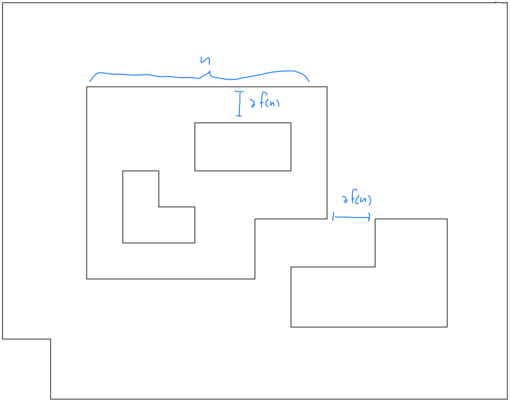


Two tools:

Definition

An f-toast on a graph G is a family of sets $\mathcal{T} \subseteq \mathcal{P}(V)$ which is

1. Finitary: Each piece $A \in \mathcal{T}$ is finite
2. Complete: $\bigcup \mathcal{T} = V$
3. Laminar: For any $A, B \in \mathcal{T}$, $A \cap B = \emptyset, A$, or B, and
4. f-separated: For $A, B \in \mathcal{T}$, $d(\partial A, \partial B) \geq f(\text{diam}(B))$.


n
≥ f(n)
≥ f(n)

Definition

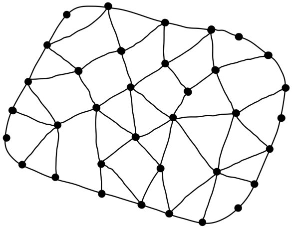
For a planar graph G, and $C \subseteq V(G)$ the boundary of a face, and $x, y \in C$ neighbors, a list-assignment L is restricted on C if

1. $|L(v)| \geq 5$ for $v \notin C$
2. $|L(v)| \geq 3$ for $v \neq x, y$
3. $|L(x)|, |L(y)| \geq 1$, and $L(x) \neq L(y)$.

Theorem (Thomassen)

Planar graphs are 5-list colorable. In fact if G is planar and L is restricted on a facial boundary of G, then G admits an L-coloring.

Thomassen's theorem



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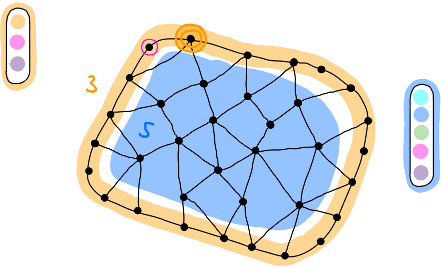
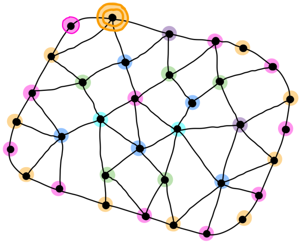
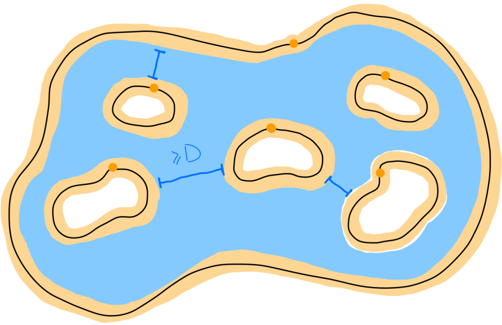


Thomassen's theorem



Theorem (Postle–Thomas)

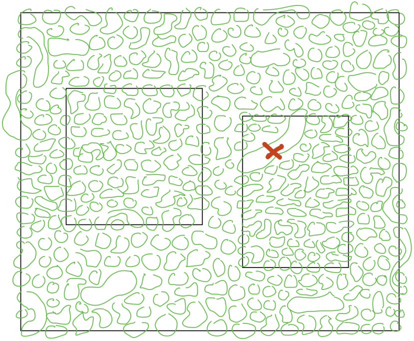
There is some D so that if G is a planar graph, $C_1, \ldots, C_n$ are facial boundaries of G with pairwise distance at least D, and L is a list assignment that is restricted on each C_i, then G admits an L-coloring.



Proof sketch of the subcritical case

Lemma

There is fiid $100D$ log-separated toast on $\mathbb{Z}^2$ so that each cell of diameter n only touches G_p-components of diameter $\log(n)$



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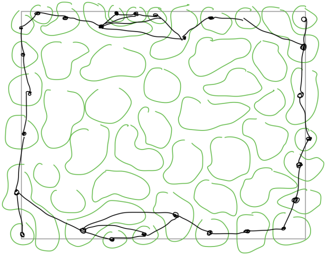

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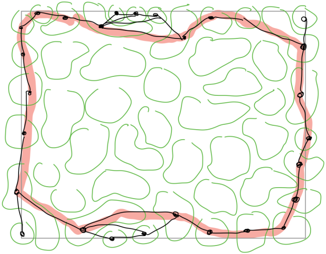

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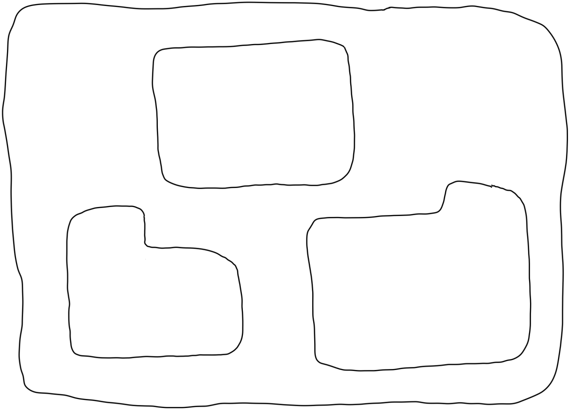

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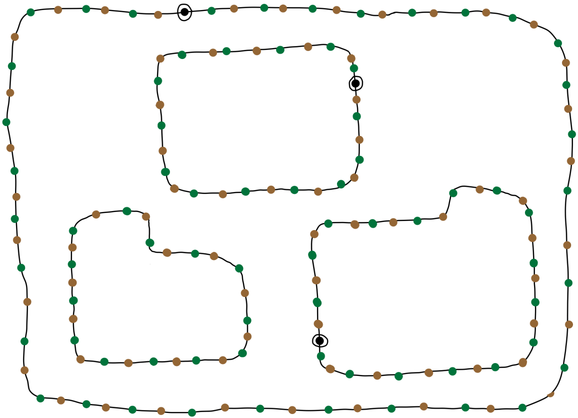

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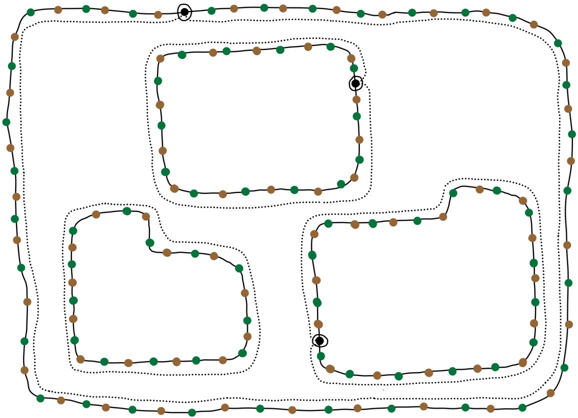

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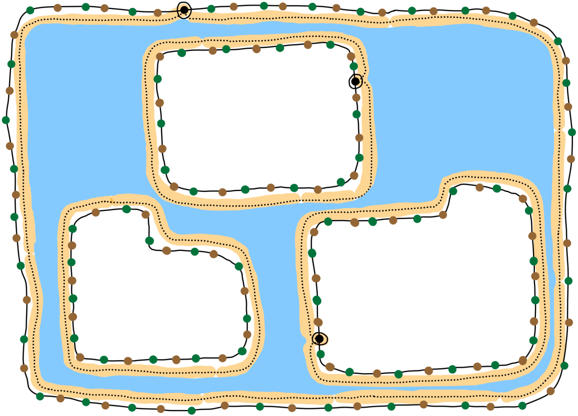


Bonus theorems

Theorem

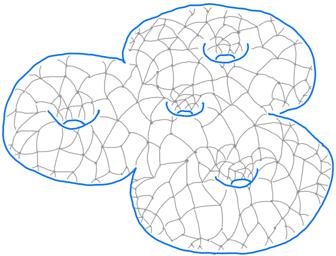
If G is Borel planar and admits connected Borel toast with 2-regular topological boundaries, G is Borel 5-colorable.

We are in the process of checking that such a nice toast can be found on every bounded degree hyperfinite planar pmp graph.

Bonus theorems

Theorem (Thomassen)

For any surface S, if G is a graph on S with no small non-contractible cycles, then G is 5-colorable.



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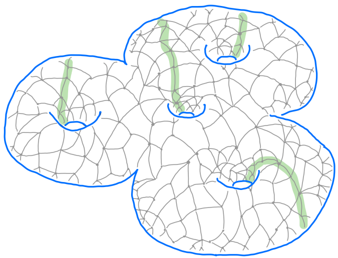

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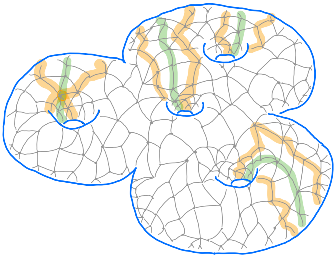

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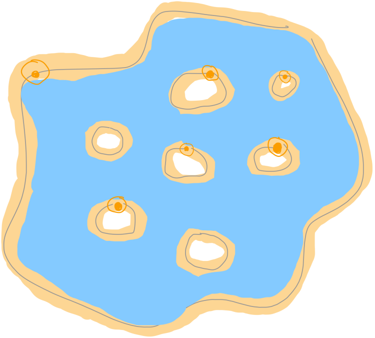

Some further questions:

1. Is $\widetilde{\mathbb{Z}^2}_p$ 4-colorable?
2. Can we 5-color $\tilde{G}_p$ for any unimodular amenable planar G?
3. For general (not necessarily planar) graphs G, what does $\tilde{G}_p$ look like when $p > p_u$?
4. How does $\chi_\mu(\tilde{G}_p)$ depend on p for $G = T_d$?
5. What about graphs with more general forbidden minors?

Thanks!