Maximal unfriendliness of back-and-forth relations

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Throughout the talk:

- Languages $\mathcal{L}$ are finite and relational,
- $\mathcal{L}$-structures are countable, with domain $\mathbb{N}$ where necessary,
- $\mathrm{Mod}(\mathcal{L})$ is the Polish space of countable $\mathcal{L}$-structures with domain $\mathbb{N}$.
- A structure is computable if it has domain $\mathbb{N}$ and all of its relations are computable on $\mathbb{N}$.

In $\mathrm{Mod}(\mathcal{L})$ the infinitary logic $\mathcal{L}_{\omega_1\omega}$ plays a central role.

$\mathcal{L}_{\omega_1\omega}$ allows countable conjunctions $\bigwedge\!\!\!\bigwedge$ and disjunctions $\bigvee\!\!\!\bigvee$ but only finitely many free variables.

Theorem (Lopez-Escobar)

An invariant set $A \subseteq \mathrm{Mod}(\mathcal{L})$ is Borel if and only if it is $\mathcal{L}_{\omega_1\omega}$-definable.

Define the quantifier complexity hierarchy for $\mathcal{L}_{\omega_1\omega}$:

- $\Sigma_0 = \Pi_0$: quantifier-free formulas,
- Σ_α: countable disjunctions $\bigvee\!\!\!\bigvee_i \exists \bar{y}_i\, \varphi_i$ with each φ_i a Π_{β_i} formula, $\beta_i < \alpha$,
- Π_α: countable conjunctions $\bigwedge\!\!\!\bigwedge_i \forall \bar{y}_i\, \varphi_i$ with each φ_i a Σ_{β_i} formula, $\beta_i < \alpha$.

Theorem (Vaught)

An invariant set $A \subseteq \mathrm{Mod}(\mathcal{L})$ is $\mathbf{\Pi}^0_\alpha$ if and only if it is definable by a Π_α sentence of $\mathcal{L}_{\omega_1\omega}$.

A Scott sentence for a countable $\mathcal{L}$-structure $\mathcal{A}$ is a sentence $\varphi \in \mathcal{L}_{\omega_1\omega}$ such that for every countable $\mathcal{L}$-structure $\mathcal{B}$,

$$\mathcal{B} \models \varphi \iff \mathcal{B} \cong \mathcal{A}.$$

Theorem (Scott's isomorphism theorem)

Every countable $\mathcal{L}$-structure has a Scott sentence.

Equivalently (via Lopez-Escobar): for every countable $\mathcal{A}$, the orbit

$$\mathrm{Orb}(\mathcal{A}) = \{\mathcal{B} \in \mathsf{Mod}(\mathcal{L}) : \mathcal{B} \cong \mathcal{A}\}$$

is a Borel subset of $\mathsf{Mod}(\mathcal{L})$.

Scott's proof used the back-and-forth relations. These have many different variations, but we use the standard asymmetric back-and-forth relations.

Definition

Given $\bar{a} \in \mathcal{A}$, $\bar{b} \in \mathcal{B}$, and a countable ordinal α, we define $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ by:

1. $(\mathcal{A}, \bar{a}) \leq_0 (\mathcal{B}, \bar{b})$ if $\bar{a}$ and $\bar{b}$ satisfy the same atomic formulas.
2. $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ if for every $\beta < \alpha$ and every $\bar{b}' \in \mathcal{B}$ there is $\bar{a}' \in \mathcal{A}$ such that

$$(\mathcal{B}, \bar{b}\bar{b}') \leq_\beta (\mathcal{A}, \bar{a}\bar{a}').$$

Write $(\mathcal{A}, \bar{a}) \equiv_\alpha (\mathcal{B}, \bar{b})$ if $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ and $(\mathcal{B}, \bar{b}) \leq_\alpha (\mathcal{A}, \bar{a})$.

Theorem (Karp, Lopez-Escobar, Vaught)

$$(\mathcal{A},\bar{a}) \leq_\alpha (\mathcal{B},\bar{b}) \Leftrightarrow \textit{every } \Pi_\alpha \textit{ formula true of } \bar{a} \textit{ in } \mathcal{A} \textit{ is true of } \bar{b} \textit{ in } \mathcal{B}$$
$$\Leftrightarrow \textit{every } \Sigma_\alpha \textit{ formula true of } \bar{b} \textit{ in } \mathcal{B} \textit{ is true of } \bar{a} \textit{ in } \mathcal{A}.$$

$$\mathcal{A} \leq_\alpha \mathcal{B} \Leftrightarrow \textit{every } \Pi_\alpha \textit{ sentence true in } \mathcal{A} \textit{ is true in } \mathcal{B}$$
$$\Leftrightarrow \textit{every } \Sigma_\alpha \textit{ sentence true in } \mathcal{B} \textit{ is true in } \mathcal{A}$$
$$\Leftrightarrow \textit{no } \boldsymbol{\Pi}^0_\alpha \textit{ set contains } \mathrm{Orb}(\mathcal{A}) \textit{ but avoids } \mathrm{Orb}(\mathcal{B})$$
$$\Leftrightarrow \textit{no } \boldsymbol{\Sigma}^0_\alpha \textit{ set contains } \mathrm{Orb}(\mathcal{B}) \textit{ but avoids } \mathrm{Orb}(\mathcal{A}).$$

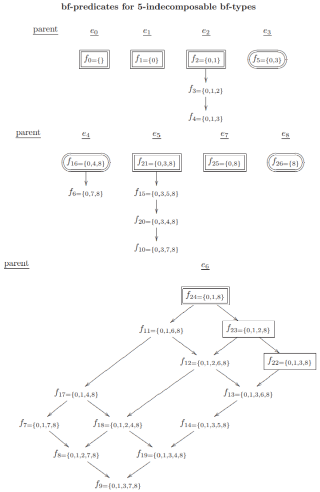
Back-and-forth relations are a key tool for countable structures.

- Scott's isomorphism theorem.
- Morley's theorem that a theory has either $\aleph_0$, $\aleph_1$, or $2^{\aleph_0}$-many models.
- Montalbán's theorem that being a counterexample to Vaught's conjecture is the same as non-trivially satisfying hyperarithmetic-is-recursive on a cone.

For Boolean algebras:

- Ketonen's theorem that there is a Boolean algebra B with $B \oplus B \oplus B \cong B$ but $B \oplus B \ncong B$ used Ketonen's invariants which are a variation of back-and-forth relations.
- Downey, Jockusch, Thurber, Knight, Stob: any Boolean algebra B with $0'''' \geq_T B''''$ (low_4) has a computable copy — the proof requires understanding the relations $\leq_4$ on Boolean algebras.
- Harris and Montalban show that new behaviour in the 5-back-and-forth types makes this problem much harder for low_5 Boolean algebras.

The (basic) 5-back-and-forth types, from *On the n-back-and-forth types of Boolean algebras* by Harris and Montalbán:



Any application of true stages in computable structure theory uses back-and-forth relations.[1]

Theorem (Knight)

Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be α-friendly computable structures. Then $\mathcal{A}_0 \leq_\alpha \mathcal{A}_1$ if and only if for every Σ^0_α set $S \subseteq \omega$ there is a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ with

$$\mathcal{C}_n \cong \begin{cases} \mathcal{A}_1 & \text{if } n \in S, \\ \mathcal{A}_0 & \text{if } n \notin S. \end{cases}$$

Definition

Computable structures $\mathcal{A}_0, \mathcal{A}_1$ are α-friendly if the relations $\leq_\beta$ on tuples of $\mathcal{A}_0$ and $\mathcal{A}_1$ are uniformly c.e. for each $\beta < \alpha$.

[1] A boldface DST version of this can be obtained by Louveau and Saint Raymond's separation theorem plus Vaught transforms.

Suppose I want to remove the assumption of α-friendliness from this theorem.

Question

What oracles X make the following fact true?

Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be computable structures. Suppose that $\mathcal{A}_0 \leq_\alpha \mathcal{A}_1$. Given a $\Sigma^0_\alpha(X)$ set $S \subseteq \omega$ there is a uniformly X-computable sequence $(\mathcal{C}_n)_{n\in\omega}$ with

$$\mathcal{C}_n \cong \begin{cases} \mathcal{A}_1 & \text{if } n \in S, \\ \mathcal{A}_0 & \text{if } n \notin S. \end{cases}$$

Each play of the back-and-forth game takes two quantifiers:

$$\begin{aligned}(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b}) &\iff \forall \bar{b}' \, \exists \bar{a}' \, (\mathcal{B}, \bar{b}\bar{b}') \leq_{\alpha-1} (\mathcal{A}, \bar{a}\bar{a}') \\ &\iff \forall \bar{b}' \, \exists \bar{a}' \, \forall \bar{a}'' \, \exists \bar{b}'' \, (\mathcal{A}, \bar{a}\bar{a}'\bar{a}'') \leq_{\alpha-2} (\mathcal{B}, \bar{b}\bar{b}'\bar{b}'') \\ &\iff \cdots\end{aligned}$$

This gives a $\Pi_{2\alpha}$ definition of $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$.

Thus:

- $\{(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b}) \mid (\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})\}$ is (lightface) $\Pi^0_{2\alpha}$.
- Given $\mathcal{A}$ computable, $\leq_\alpha$ is $\Pi^0_{2\alpha}$, hence $0^{(2n)}/0^{(2\alpha+1)}$-computable.

Fact

Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be computable structures and let $X \geq_T 0^{(2\alpha+1)}$. Suppose that $\mathcal{A}_0 \leq_\alpha \mathcal{A}_1$. Given a $\Sigma^0_\alpha(X)$ set $S \subseteq \omega$ there is a uniformly X-computable sequence $(\mathcal{C}_n)_{n\in\omega}$ with

$$\mathcal{C}_n \cong \begin{cases} \mathcal{A}_1 & \text{if } n \in S, \\ \mathcal{A}_0 & \text{if } n \notin S. \end{cases}$$

Fact

If $\mathrm{Orb}(\mathcal{A})$ *is* $\mathbf{\Pi}^0_\alpha$ *for* $\alpha < \omega_1^{ck}$*, then it is (lightface)* $\Pi^0_{2\alpha}$*.*

In particular, if $\mathcal{A}$ *has computable Scott rank then it has a computable Scott sentence.*

Proof.

If $\mathrm{Orb}(\mathcal{A})$ is $\mathbf{\Pi}^0_\alpha$, then $\mathcal{A}$ has a Π_α Scott sentence. Thus

$$\mathrm{Orb}(\mathcal{A}) = \{\mathcal{B} \in \mathsf{Mod}(\mathcal{L}) : \mathcal{A} \leq_\alpha \mathcal{B}\}.$$

This is (lightface) $\Pi^0_{2\alpha}(\mathcal{A})$. □

Fact

Suppose that T is a Π_2 sentence, and in the Polish space $\mathrm{Mod}(T)$, *each orbit is $\mathbf{\Pi}^0_\alpha$ for $\alpha < \omega_1^{ck}$. Then isomorphism on T is (lightface) $\Pi^0_{2\alpha}$.*

Proof.

Each model of T has a Π_α Scott sentence. Then for models of T,

$$\mathcal{A} \equiv_\alpha \mathcal{B} \Longleftrightarrow \mathcal{A} \cong \mathcal{B}.$$

But $\equiv_\alpha$ is $\Pi^0_{2\alpha}$. □

Fact

Given $\bar{a} \in \mathcal{A}$*, there is a* Π_α *formula* $\varphi_{\bar{a}}(\bar{x})$ *such that*

$$\mathcal{A} \models \varphi_{\bar{a}}(\bar{b}) \iff (\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{A}, \bar{b}).$$

What about across different structures?

Fact

Given $\bar{a} \in \mathcal{A}$*, there is a* $\Pi_{2\alpha}$ *formula* $\varphi_{\bar{a}}(\bar{x})$ *such that*

$$\mathcal{B} \models \varphi_{\bar{a}}(\bar{b}) \iff (\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b}).$$

Question

Can any of these be improved?

In general, this upper bound is tight.

Theorem (Chen, Gonzalez, Harrison-Trainor)

The set

$$\{(\mathcal{A}, \mathcal{B}) : \mathcal{A} \leq_\alpha \mathcal{B}\}$$

is $\mathbf{\Pi}^0_{2\alpha}$*-complete.*

Thus $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ has no definition simpler than $\Pi_{2\alpha}$.

But that doesn't mean that this is tight in all applications!

Theorem (Chen, Gonzalez, HT)

Given $\bar{a} \in \mathcal{A}$, there is a $\Pi_{\alpha+2}$ formula $\varphi_{\bar{a}}(\bar{x})$ such that

$$\mathcal{B} \models \varphi_{\bar{a}}(\bar{b}) \iff (\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$$

and a $\Pi_{\alpha+3}$ formula $\psi_{\bar{a}}(\bar{x})$ such that

$$\mathcal{B} \models \psi_{\bar{a}}(\bar{b}) \iff (\mathcal{B}, \bar{b}) \leq_\alpha (\mathcal{A}, \bar{a})$$

For most α these cannot be improved further.

Question

This is not effective, i.e., if $\mathcal{A}$ is computable, then the formulas $\varphi_{\bar{a}}$ and $\psi_{\bar{a}}$ are not computable.

What is the best one can do with computable formulas?

Fact

If a computable structure has a Π_α Scott sentence, then it has a computable $\Pi_{2\alpha}$ Scott sentence.

Proof.

If $\mathcal{A}$ has a Π_α Scott sentence then

$$\mathcal{A} \leq_\alpha \mathcal{B} \iff \mathcal{A} \cong \mathcal{B}$$

and there is a computable $\Pi_{2\alpha}$ sentence φ such that

$$\mathcal{A} \leq_\alpha \mathcal{B} \iff \mathcal{B} \models \varphi. \qquad \square$$

Theorem (Alvir, Csima, Harrison-Trainor)

There is a computable structure $\mathcal{A}$ with a Π_2 Scott sentence but no computable Σ_4 such sentence.

There is a technique called Marker extensions.

Consider a graph G. We can form a graph G^* by replacing each edge with an elaborate gadget, and each non-edge by a different elaborate gadget.

For example, we might replace each edge by a copy of ω, and each non-edge by a copy of ω^*. Then:

- A Σ_n/Π_n fact about G becomes a Σ_{n+1}/Π_{n+1} fact about G^*.
- There is a continuous map taking G to G^*, and a Δ^0_2-measurable map taking G^* to G.
- G has an X'-computable copy if and only if G^* has an X-computable copy.

More generally there are gadgets that make Σ_α/Π_α facts about G become $\Sigma_{\alpha+\beta}/\Pi_{\alpha+\beta}$ facts about G^*.

A change of topology is like Morleyizing: We take a formula $\varphi(\bar{x})$ and name it with a new relation R, making $\{\mathcal{A} : \mathcal{A} \models \varphi(\bar{a})\}$ clopen.

A Marker extension is the reverse of this: We take a relation $R(\bar{x})$ and we remove it from the language, replacing it by a formula $\varphi(\bar{x})$.

Like a reverse change of topology: The space of (ω, ω^*)-Marker extensions of graphs is like the space of graphs where the edge relation gives basic Δ^0_2 sets.

Marker extensions can be used to "bump up" the complexity of an example.

Theorem (Alvir, Csima, Harrison-Trainor)

There is a computable structure with a Π_2 Scott sentence but no computable Σ_4 Scott sentence.

Corollary (Alvir, Csima, Harrison-Trainor)

There is a computable structure with a Π_n Scott sentence but no computable Σ_{n+2} Scott sentence.

Marker extensions work additively.

The reason is that the gadgets are friendly, e.g.:

$$\omega \leq_1 \omega^*, \quad \omega^* \leq_1 \omega, \quad \omega \not\leq_2 \omega^*, \quad \omega^* \not\leq_2 \omega$$

where the failures of $\leq_2$ are witnessed by computable formulas.

Question

What if we used gadgets where $\not\leq_n$ is not witnessed by computable formulas?

I introduced unfriendly Marker extensions as a type of Marker extension where the gadgets are as unfriendly as possible, i.e., the back-and-forth relations are as complicated as possible.

The gadgets must be constructed by hand and construction is recursive: To show that maximally unfriendly gadgets exist at a certain level, we use unfriendly jump inversions at the previous level.

You should think of this like a reverse change-of-topology where the boldface and lightface complexities do not agree.

For a graph $\mathcal{G}$, write $\mathcal{G}^{(-n)}$ for the nth jump inversion of $\mathcal{G}$, i.e., the Marker extension making the edge relation lightface and boldface Δ^0_{n+1}.

Write $\mathcal{G}_u^{(-n)}$ for the nth unfriendly jump inversion of $\mathcal{G}$, i.e., the unfriendly Marker extension making the edge relation boldface Δ^0_{n+1} but lightface Δ^0_{2n+1}.

For friendly jump inversions:

1. Given a $\mathbf{d}$-computable copy of $\mathcal{G}^{(-n)}$ there is a $\mathbf{d}^{(n)}$-computable copy of $\mathcal{G}$.
2. Given a $\mathbf{d}^{(n)}$-computable copy of $\mathcal{G}$ there is a $\mathbf{d}$-computable copy of $\mathcal{G}^{(-n)}$.

For unfriendly jump inversions:

1. Given a $\mathbf{0}^{(n)}$-computable copy of $\mathcal{G}_u^{(-n)}$, there is a $\mathbf{0}^{(2n)}$-computable copy of $\mathcal{G}$.
2. Given a $\mathbf{0}^{(2n)}$-computable copy of $\mathcal{G}$, there is a computable copy of $\mathcal{G}_u^{(-n)}$.

Unfriendly jump inversions let us show that the $n/2n$ bounds are tight for computability-related problems.

Theorem (Alvir, Csima, HT)

There is a computable structure with a Π_n Scott sentence but no computable Σ_{n+2} Scott sentence.

Theorem (HT)

There is a computable structure with a Π_n Scott sentence but no computable Σ_{2n} Scott sentence.

Marker extensions are a widely useful tool.

In a language with many relation symbols, you can make different symbols have different complexities.

This lets you make a continuous / computable construction while approximating whether some relation is true, e.g., if we make R Δ^0_2 then in a computable construction we give a limit approximation of R.

Unfriendly Marker extensions make the type of approximation different depending on oracle, e.g.:

- In a computable construction we can give a Δ^0_3 approximation of R.
- In a $0'$-computable construction we must give a Δ^0_2 approximation of R.

But these are the same thing! $0'$ gets no additional power.

Theorem (Slaman, Wehner, 1998)

There is a structure which has no computable copy, but which can be computed from any non-computable oracle.

Using Marker extensions:

Theorem (Greenberg, Montalbán, Slaman, 2013)

There is a structure which has no hyperarithmetic copy, but which can be computed from any non-hyperarithmetic oracle.

Using unfriendly Marker extensions:

Theorem (HT, 2025)

There is a structure which has no arithmetic copy, but which can be computed from any non-arithmetic oracle.

Consider a sentence T, and the standard Borel space $\mathrm{Mod}(T)$ of models of T.

Isomorphism $\cong_T$ is an equivalence relation on $\mathrm{Mod}(T)$ and a subset of $\mathrm{Mod}(T) \times \mathrm{Mod}(T)$.

Hjorth, Kechris, Louveau: If $\cong_T$ is Borel reducible to $\cong_S$, and $\cong_S$ is potentially $\mathbf{\Pi}^0_\alpha$, then so is $\cong_T$. Thus we should study the potential Borel complexity of isomorphism.

Theorem (Hjorth, Kechris, Louveau)

The possible potential Borel complexities of isomorphism are exactly $\mathbf{\Delta}^0_1$, $\mathbf{\Pi}^0_1$, $\mathbf{\Sigma}^0_2$, $\mathbf{\Pi}^0_n$, $D(\mathbf{\Pi}^0_n)$ *(*$n \geq 3$*)*, $\bigoplus_{\alpha<\lambda} \mathbf{\Pi}^0_\alpha$, $\mathbf{\Pi}^0_\lambda$, $\mathbf{\Sigma}^0_{\lambda+1}$, $\mathbf{\Pi}^0_{\lambda+n}$, $D(\mathbf{\Pi}^0_{\lambda+n})$ *(*λ *limit,* $n \geq 2$*).*

Suppose we want to study continuous reducibility on Mod(T) where T is a Π_2 theory. Then we should consider actual Borel complexity of isomorphism.

Moreover: If Φ is a continuous reduction from $\cong_T$ to $\cong_S$, and $\Phi(\mathcal{A})$ has a Π_α Scott sentence, then so does $\mathcal{A}$. (Knight, Millar, Vanden Boom.)

We should also consider the Scott sentence complexities of models.

The Borel complexity of isomorphism and the Scott complexity of models are related.

Theorem (Calderoni, HT; Gonzalez, Turetsky)

Suppose that $\mathcal{A}$ and $\mathcal{B}$ have $\Pi_{\alpha+1}$ Scott sentences, and that $\mathcal{A} \equiv_\alpha \mathcal{B}$. Then $\mathcal{A} \cong \mathcal{B}$.

Thus if all models have $\Pi_{\alpha+1}$ Scott sentences, isomorphism is $\Pi^0_{2\alpha}$.

Theorem (Calderoni, HT)

All combinations n, m not previously excluded of

- *isomorphism on T is $\mathbf{\Pi}^0_n$-complete*

and

- *each model of T has a Π_m Scott sentence and no model of T has a Σ_m Scott sentence*

are realized by Π_2 theories T.

Moreover, there are Π_2 theories where isomorphism on T is $\mathbf{\Sigma}^0_n$-complete, and there are many possible Scott sentence complexities.

Thank you for your attention.