# Condensed sets and the Solovay model

Nathaniel Bannister

Carnegie Mellon University

May 6, 2026

Joint work with Dianthe Basak

# Condensed Sets

Let $\kappa$ be a inaccessible cardinal.

**Definition**

*A $\kappa$-condensed/pyknotic set/ring/group... is a functor*

$$F\colon \{\textit{profinite sets of size } {<}\kappa\}^{op} \to \textit{sets/rings/groups...}$$

*satisfying*

- $F(\varnothing) = *$
- $F(S \coprod T) \cong F(S) \times F(T)$
- *If $f : S \to T$ is a continuous surjection of profinite sets with fiber product $S \times_T S$ and projection maps $p_1, p_2$, the map*

  $$F(f)\colon F(T) \to \{x \in F(S) \mid F(p_1)(x) = F(p_2)(x)\}$$

  *is a bijection.*

# Extremally disconnected spaces

### Example

*If $X$ is a space, $\underline{X}(S) = C(S, X)$ is a $\kappa$-pyknotic set for every $\kappa$.*

Set theory starts to enter the picture through *extremally disconnected* compact Hausdorff space.

### Theorem (Gleason)

*The following are equivalent for a compact Hausdorff space $X$:*

- *For all open $U \subseteq X$, $\overline{U}$ is clopen;*
- *$X$ is projective in the category of compact Hausdorff spaces;*

$$\begin{array}{ccc} & & Y \\ & \nearrow & \big\downarrow \\ X & \longrightarrow & Z \end{array}$$

- *$X$ is homeomorphic to the Stone space of a complete Boolean algebra*

# Extremally disconnected spaces and condensed sets

**Fact**

*A condensed set is (essentially) determined by its restriction to the extremally disconnected compact Hausdorff spaces. Moreover, $\kappa$-pyknotic sets are equivalent to the functors*

$$F\colon \{\textit{edcH spaces of size} <\kappa\}^{op} \to \textit{Sets}$$

*such that*

- $F(\varnothing) = *$
- $F(S \coprod T) = F(S) \times F(T)$.

# Forcing and condensed sets

Theorem (Bergfalk, Lambie-Hanson, Šaroch)

*For $X$ a Hausdorff space and $\mathbb{B}$ a complete Boolean algebra, $\underline{X}(S(\mathbb{B}))$ corresponds to "names for points in a compact reinterprertation of $X$" and this is useful for condensed mathematics.*

For instance, they give a forcing proof of

Theorem (Clausen, Scholze)

*If $A$ is a discrete abelian group of size $<\kappa$ and*

$$\underline{Ext}^1(\underline{A}, \underline{\mathbb{Z}}) = 0$$

*in* **Pyk**, *then $A$ is free.*

# Can we get a more global picture?

**Definition**

*For $\kappa$ inaccessible, the Solovay model is $V(\mathbb{R})^{V^{\mathrm{Coll}(\omega,<\kappa)}}$, the smallest model of ZF containing all ground model sets and all new reals numbers after collapsing $\kappa$ to become $\aleph_1$. Equivalently, the Solovay model consists of all sets in $V^{\mathrm{Coll}(\omega,<\kappa)}$ which are hereditarily definable from ground model and real parameters.*

We will build a functor $\Lambda$ from the category of **Pyk** to the category **VD** of $V$-definable sets in the Solovay model and $V$-definable functions.

# The category **VD**

Officially, the functor $\Lambda$ exists in $V$ and the category **VD** consists of definitions for objects in the Solovay model and definitions for maps between them.

Definition

*We define* **VD** *to be the category where*

- *Objects are* $\mathbb{K}$*-names* $\dot{a}$ *such that*
  - *for some parameter* $b$ *and formula* $\psi(x, y)$*,*

    $$\Vdash_{\mathbb{K}} \psi(x, \check{b}) \text{ defines } \dot{a}$$

  - $\Vdash_{\mathbb{K}} \dot{a} \in V(\mathbb{R})$.
- *A morphism from* $\dot{a}$ *to* $\dot{b}$ *is a name* $\dot{f}$ *for a function from* $\dot{a}$ *to* $\dot{b}$ *such that for some formula* $\psi(x, y)$ *and parameter* $c \in V$*,* $\Vdash \psi(x, \check{c})$ *defines* $\dot{f}$*. We identify two names for functions if they are forced to be equal.*

# The functor Λ

The functor Λ maps a condensed set $F$ to "the filtered colimit of the values of $F$ over all ways to force with all small forcings,"

$$\operatorname*{colim}_{\substack{\mathbb{B}\in V_\kappa \text{ cBa} \\ I\subseteq\mathbb{B}\ V-\text{generic}}} F(S(\mathbb{B})).$$

**Definition**

*$\Lambda(F)$ is a name for the quotient of triples $(x, \mathbb{B}, I)$ where*

- *$\mathbb{B} \in V_\kappa$ is a complete Boolean algebra in the sense of $V$;*
- *$x \in F(S(\mathbb{B}))$;*
- *$I$ is a $V$-generic filter on $\mathbb{B}$*

*and $(x, \mathbb{B}, I) \simeq (y, \mathbb{C}, J)$ if and only if there is a complete Boolean algebra $\mathbb{D}$, a generic filter $K$ on $\mathbb{D}$, and ground model complete Boolean homomorphisms $i_1 : \mathbb{B} \to \mathbb{D}$ and $i_2 : \mathbb{C} \to \mathbb{D}$ such that $I = i_1^{-1}K, J = i_2^{-1}K$, and $F(S(i_1))(x) = F(S(i_2))(y)$.*

# Some example computations

## Proposition

*If $X$ is a Hausdorff space, then $\Lambda(\underline{X})$ is naturally isomorphic to the set of maximal filters on $K(X)^V$, the poset of ground model compact subsets of $X$, which are in the Solovay model $V(\mathbb{R})$. The isomorphism maps $(f, \mathbb{B}, I)$ to*

$$\{C \mid \exists b \in I(N_b \subseteq f^{-1}C)\}$$

## Corollary

- *Suppose $X \subseteq [0,1]^I$ is closed. $\Lambda(\underline{X})$ is isomorphic to the underlying set of $\overline{\check{X}} \subseteq [0,1]^I$.*
- $\Lambda\left(\underline{\prod_I \mathbb{Z}}\right)$ *is isomorphic to $\prod_I^{bdd} \mathbb{Z}$, defined as*

$$\left\{ e \in \prod_I \mathbb{Z} \;\middle|\; \exists f : I \to \omega(f \in V \wedge \forall i(|e(i)| \leq f(i))) \right\}$$

# Factoring $\Lambda$: the sites **Sol** and **Sol**$_{\neg\neg}$

The functor $\Lambda$ factors through several sheaf categories, in a way that allows us to shed light on its nature. The first step along the way is the site **Sol**, which retains only the duals of complete Boolean homomorhpisms.

Definition

*The site* **Sol** *has*

- **Objects:** *Extremally disconnected Stone spaces of size* $< \kappa$.
- **Morphisms: Open** *continuous maps with composition of maps.*
- **Covers: Finite** *collections of maps* $\{f_i \colon X_i \to X\}$ *whose images jointly cover* $X$.

**Sol**$_{\neg\neg}$ *is the same category as above but with*

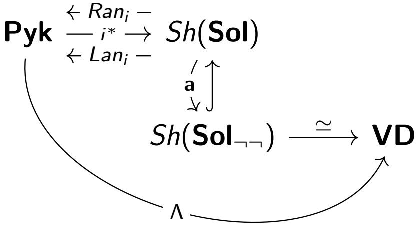
- **Covers: Arbitrary** *collections of maps* $\{f_i \colon X_i \to X\}$ *such that* $\bigcup_i f_i[X_i]$ *is* **dense** *in* $X$.

*This is also called the* **dense coverage** *on the above category.*

# Factoring Λ

**Theorem (B., Basak)**

*There is a diagram*



# On the Whitehead problem

We now sketch a proof of:

**Theorem (Clausen, Scholze)**

*If $A$ is a discrete abelian group of size $<\kappa$ and*

$$\underline{Ext}^1(\underline{A}, \underline{\mathbb{Z}}) = 0$$

*in* **Pyk**, *then $A$ is free.*

Fix a free resolution

$$0 \to \bigoplus_I \mathbb{Z} \to \bigoplus_J \mathbb{Z} \to A \to 0.$$

$\underline{Ext}^1(\underline{A}, \underline{\mathbb{Z}}) = 0$ implies that

$$0 \to \underline{Hom}(\underline{A}, \underline{\mathbb{Z}}) \to \prod_J \underline{\mathbb{Z}} \to \prod_I \underline{\mathbb{Z}} \to 0$$

is exact in **Pyk**.

# On the Whitehead problem (cont.)

So

$$0 \to \Lambda\left(\underline{Hom}(\underline{A}, \underline{\mathbb{Z}})\right) \to \Lambda\left(\prod_J \underline{\mathbb{Z}}\right) \to \Lambda\left(\prod_I \underline{\mathbb{Z}}\right) \to 0$$

is an exact sequence in the Solovay model. Therefore,

$$\pi\colon \prod_J^{bdd} \mathbb{Z} \to \prod_I^{bdd} \mathbb{Z}$$

is a surjective group homomorphism. It is also continuous when these groups are viewed as topological subgroups of their respective products. By the Baire category theorem, whenever $e\colon I \to \omega$ is in $V$, there is an $f\colon J \to \omega$ in $V$ such that

$$\pi\left(\prod(f)\right) \supseteq \prod(e).$$

# On the Whitehead problem (cont.)

Using a theorem of Nöbeling, we now assume (in $V$) that $\bigoplus_I \mathbb{Z} \cong C(S, \mathbb{Z})$ for some extremally disconnected compact Hausdorff space $S$. Then there is a natural continuous embedding

$$j\colon S \to \prod_I \mathbb{Z}.$$

By compactness, there is $e\colon I \to \omega$ such that $j[S] \subseteq \prod(e)$. Since $S$ is extremally disconnected, there is a section

$$\begin{array}{ccccc} & & \prod(f) \cap \pi^{-1}\prod(e) & \hookrightarrow & \prod_J \mathbb{Z} \\ & \overset{s}{\nearrow} & \Big\downarrow \pi & & \Big\downarrow \pi \\ S & \longrightarrow & \prod(e) & \hookrightarrow & \prod_I \mathbb{Z} \end{array}$$

# On the Whitehead problem

For each $j \in J$, projection onto the $j$th coordinate yields a continuous function from $S$ to $\mathbb{Z}$ and therefore an element of $\bigoplus_I \mathbb{Z}$. Assembling these elements yields a homomorphism $h\colon \bigoplus_J \mathbb{Z} \to \bigoplus_I \mathbb{Z}$, the dual of which extends $s$. Using linear independence of the basis vectors in $C(S, \mathbb{Z}) \cong \bigoplus_I \mathbb{Z}$, we then argue that any two continuous homomorphisms from $\prod_I \mathbb{Z}$ to $\mathbb{Z}$ which coincide on $S$ must coincide. And consequently that $h$ splits the exact sequence

$$0 \to \bigoplus_I \mathbb{Z} \to \bigoplus_J \mathbb{Z} \to A \to 0.$$

In particular, $A$ is a subgroup of a free group and therefore free.

Thank you!