A dynamical proof of Matui's absorption theorem

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Caltech Logic Seminar

I. Some background.

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The associated equivalence relation is obtained from E_0 by gluing the classes of 0^∞ and 1^∞ together, where

$$(x \, E_0 \, y) \Leftrightarrow (\exists n \, \forall i \geq n \;\, x(i) = y(i)) \, .$$

Kakutani–Rokhlin partitions

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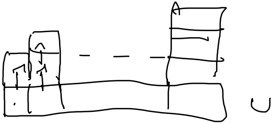
Then $n\colon U \to \omega$ is continuous, hence has finite range F and each $U_i = \{x \in U\colon n(x) = i\}$ is clopen. One has

$$X = \bigsqcup_{i \in F} \bigsqcup_{j=0}^{i-1} \varphi^j(U).$$



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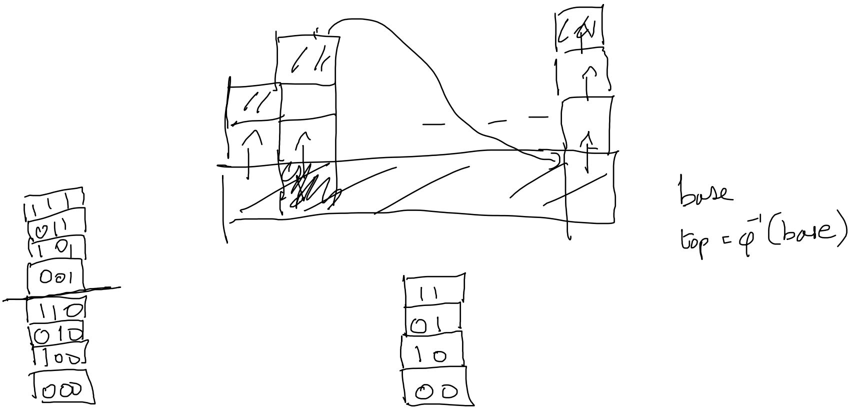
This is the archetype of a *Kakutani–Rokhlin partition*: a clopen partition $(A_{i,j})_{i \in F, j \leq n_i}$ such that $\varphi(A_{i,j}) = A_{i,j+1}$ for all $j \leq n_i - 1$.

Let us try to draw a Kakutani–Rokhlin partition



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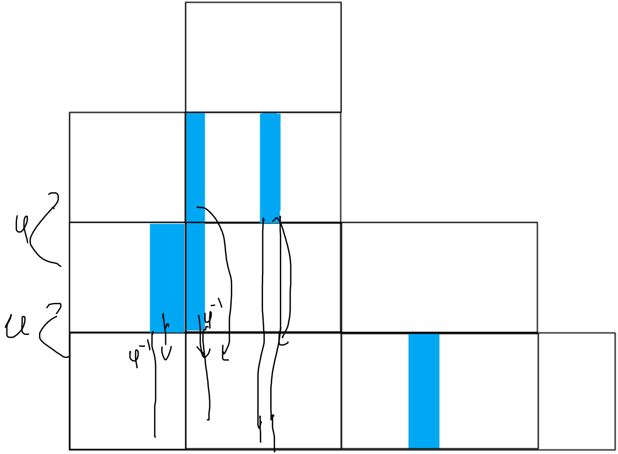
- For all n, $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$.
- The bases and tops of $\mathcal{A}_n$ each shrink to a point (x and $\varphi^{-1}(x)$).
- Given U clopen, there exists n such that U is a union of atoms of $\mathcal{A}_n$.

Cutting a Kakutani-Rokhlin partition to make it compatible with a clopen set



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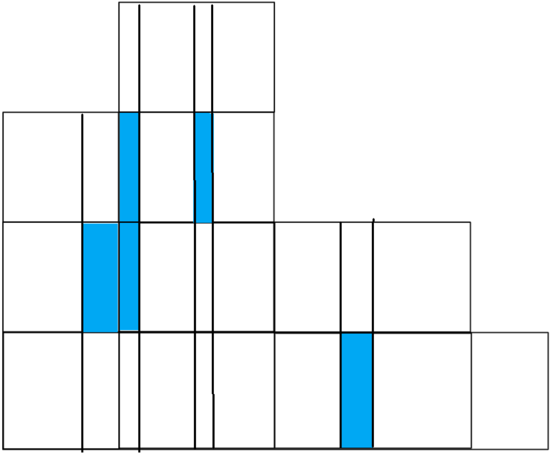

Full groups I.

Definition

A subgroup $G \leq \mathrm{Homeo}(\mathrm{X})$ is a *full group* if: whenever $U_0, \ldots, U_n$ is a clopen partition of X, $g_0, \ldots, g_n$ are elements of G, and $g \in \mathrm{Homeo}(X)$ is such that $g_{\restriction U_i} = g_{i \restriction U_i}$ for all i, then $g \in G$.

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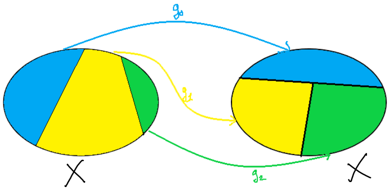

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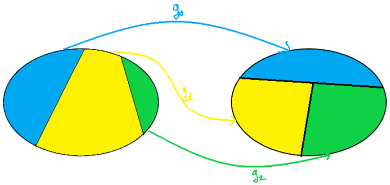

Definition

The *topological full group* of φ, denoted $[\![\varphi]\!]$, is the smallest full group containing φ. It is a countable subgroup of $\mathrm{Homeo}(X)$.

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The *full group* of φ is

$$[\varphi] = \{g \in \mathrm{Homeo}(X) \colon \forall x \; \exists n_x \quad g(x) = \varphi^{n_x}(x)\}$$

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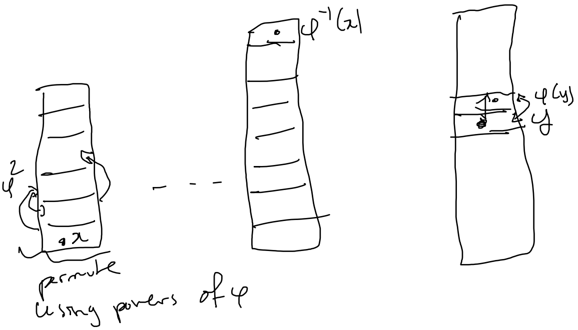
$[\![\varphi]\!]$ consists of all elements of $[\varphi]$ such that $x \mapsto n_x$ is continuous; $[\varphi]$ is an uncountable group (actually coanalytic, non-Borel).

A locally finite subgroup

Fix a minimal φ and $x \in X$. Denote $O^+(x) = \{\varphi^n(x) : n \geq 0\}$.
Define $\Gamma_x(\varphi)$ as the set of all $g \in [\![\varphi]\!]$ s.t. $g(O^+(x)) = O^+(x)$.



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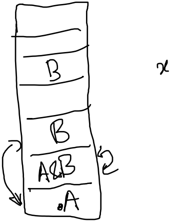
- The φ-orbit of x splits into two $\Gamma_x(\varphi)$-orbits (positive and negative half orbits): via $\Gamma_x(\varphi)$, it is not possible to move $\varphi^{-1}(x)$ to x.
- All other orbits for the actions of φ and $\Gamma_x(\varphi)$ on X coincide.

Actions on clopen sets

Theorem (Glasner–Weiss 1995)

Fix a minimal φ, and $x \in X$. Denote by $M(\varphi)$ the set of φ-invariant measures.
For any two clopen A, B such that $\mu(A) < \mu(B)$ for all $\mu \in M(\varphi)$, there exists $g \in \Gamma_x(\varphi)$ such that $g(A) \subset B$.



$M(\varphi)$ is compact

$$\frac{1}{N}\sum_{n=0}^{N-1} \delta_{\varphi^n(x)}(A) = \mu_{N,x}(A)$$

$\longrightarrow \exists N \ \forall x \ \ \mu_{N,x}(A) \leq \mu_{N,x}(B)$

$\longrightarrow$ #A in a column < #B in a column

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- For A, B clopen, there exists $g \in \Gamma_x(\varphi)$ s.t. $g(A) = B$ iff there exists $g \in [\![\varphi]\!]$ s.t. $g(A) = B$.
- In general, this is stronger than the existence of $g \in [\varphi]$ s.t. $g(A) = B$.

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Each $\Gamma_x(\varphi)$ is an ample group. (Actually, every ample group acting minimally is a $\Gamma_x(\varphi)$ for some minimal φ).

Krieger's theorem

Theorem (Krieger 1980)

Assume that Γ, Λ are two ample subgroups of $\mathrm{Homeo}(X)$ such that for any clopen U, V

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Krieger's theorem is proved by intertwining exhaustive sequences of finit unit systems for Γ, Λ.

II. A first absorption theorem.

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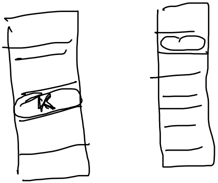
Then $\mu(K) = 0$ for all $\mu \in M(\Gamma)$

Krieger's theorem, v2.

Assume that K is Γ-sparse. Then one can build an exhaustive sequence of finite unit systems $(\mathcal{A}_n, \Gamma_n)$ such that for each atom A of $\mathcal{A}_n$ at most one element of $\Gamma_n A$ intersects K.



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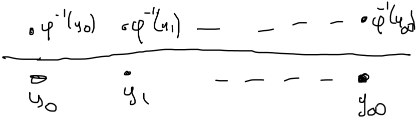
Theorem (M.-Robert 2022)

Assume that K, L are Γ-sparse and $h\colon K \to L$ is a homeomorphism. Then h extends to a homeomorphism of X such that $h\Gamma h^{-1} = \Gamma$.

Let φ be a minimal homeomorphism and $(y_n)_n$ converge to some y_∞; assume that for all $n \neq m$ y_n, y_m, y_∞ belong to different orbits. Denote $Y = \{y_n : n < \omega\} \cup \{y_\infty\}$ and $\Gamma = \bigcap_{y \in Y} \Gamma_y(\varphi)$.



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- Γ is ample; the Γ-orbit of every $y \in Y$ is its φ-orbit split into its positive and negative parts ; and all other Γ-orbits coincide with φ-orbits.
- The map h sending y_n to y_{n+1}, $\varphi^{-1}(y_n)$ to $\varphi^{-1}(y_{n+1})$ and fixing y_∞, $\varphi^{-1}(y_\infty)$ is a homeomorphism between two Γ-sparse closed subsets of X. $\longrightarrow$ extend to h $\quad h\Gamma h^{-1} = \Gamma$

$$\langle R_\Gamma \quad y_n \sim \varphi^{-1}(y_n),\ n \geq 0 \rangle = R_\varphi$$

$$\langle R_\Gamma,\ y_n \sim \varphi^{-1}(y_n),\ n \geq 1 \rangle = R_{\Gamma_{y_0}(\varphi)}$$

$$(h \times h)(R_\varphi) = \langle R_{h\Gamma h^{-1}},\ h(y_n) \sim h(\varphi^{-1}(y_n)) \rangle = \langle R_\Gamma,\ y_{n+1} \sim \varphi^{-1}(y_{n+1}) \rangle = R_{\Gamma_{y_0}(\varphi)}$$

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- The map sending y_n to y_{n+1}, $\varphi^{-1}(y_n)$ to $\varphi^{-1}(y_{n+1})$ and fixing y_∞, $\varphi^{-1}(y_\infty)$ is a homeomorphism between two Γ-sparse closed subsets of X.

It follows from v2 of Krieger's theorem that the relation obtained from R_Γ by gluing together the Γ-orbits of y and $\varphi^{-1}(y)$ for every $y \in Y$ is orbit equivalent to the relation obtained by gluing together the Γ-orbits of every y and $\varphi^{-1}(y)$ for every $y \in Y \setminus \{y_0\}$.

A first absorption theorem

We just obtained the following result.

Theorem (Giordano–Putnam–Skau 1995)

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Theorem (Giordano–Putnam–Skau 2004)

Assume that Γ is ample and acts minimally on X, and that $K = K_1 \sqcup \sigma(K_1)$ is Γ-sparse, where K_1 is clopen in K and σ is a homeomorphic involution. Then R_Γ is orbit equivalent to the relation obtained from R_Γ by gluing together the Γ-orbit of k and $\sigma(k)$ for every $k \in K$.

III. Matui's absorption theorem.

Γ-étale subsets

Definition

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- K is Γ-*étale* if for every $\gamma \in \Gamma$ and every $U \in \mathrm{Clopen}(K)$ the set $\gamma U \cap K$ is clopen in K.

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Every Γ-sparse closed subset is Γ-malleable.

Krieger's theorem, v3

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Assume that K is Γ-malleable. A finite unit system $(\mathcal{A}, \Delta)$ with $\Delta \leq \Gamma$ is K-*compatible* if for any atoms A, B of $\mathcal{A}$ which intersect K, and any $\delta \in \Delta$ s.t. $\delta(A) = B$, we have $\delta(A \cap K) = B \cap K$.

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Assume that K, L are Γ-malleable and $h \colon K \to L$ is a homeomorphism. Assume that $h\Gamma_K h^{-1} = \Gamma_L$. Then h extends to a homeomorphism of X such that $h\Gamma h^{-1} = \Gamma$.

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(This is closely related to the "Fundamental Lemma" of Giordano, Putnam and Skau)

Definition

Given a closed subset K and an ample subgroup Σ of $\mathrm{Homeo}(K)$, denote $R_\Gamma(K,\Sigma)$ the finest equivalence relation S which is coarser than R_Γ and is such that $\sigma(k)\ S\ k$ for every $k \in K$ and $\sigma \in \Sigma$.

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Matui's theorem is originally formulated in terms of étale equivalence relations but it is not hard to recover that statement from the proof of the statement given above.

Proof strategy

Intuitively : we want to show that R_Γ is obtained from some R_Λ by absorbing a prescribed small extension of R_Λ; but then it can be obtained from some R_Λ by absorbing ω times the same prescribed small extension, so absorbing this extension one more time does not change the OE class (by homogeneity, i.e. v3 of Krieger's theorem).

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The proof requires at the moment more technical ingenuity than the other arguments I described. Hopefully this can be improved...

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