# Separating complexity classes of LCL problems on grids

Katalin Berlow, Anton Bernshteyn, Clark Lyons, Felix Weilacher

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# Schreier Graphs

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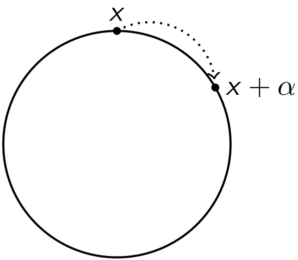

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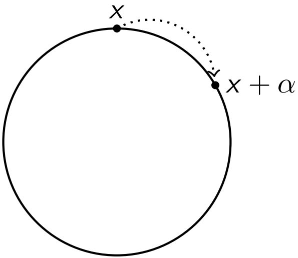


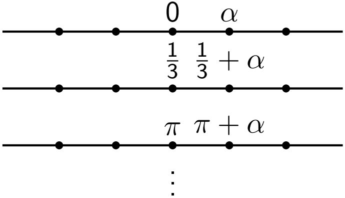

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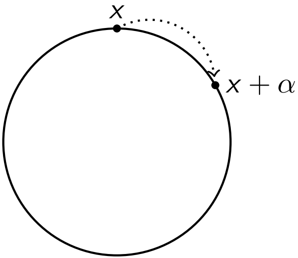


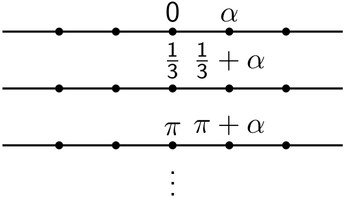


Each connected component will look like a copy of the Cayley graph of $\mathbb{Z}$ because the action is free.

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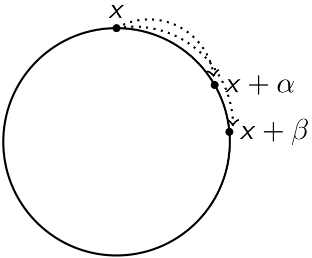
Example 2: Fix independent $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$. Take $\mathbb{Z}^2 = \langle a, b \rangle \curvearrowright \mathbb{R}/\mathbb{Z}$ by $a \cdot x = x + \alpha$ and $b \cdot x = x + \beta$.



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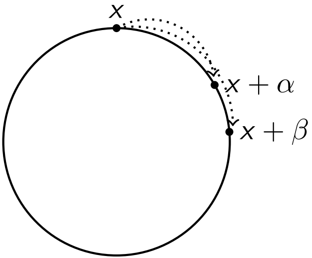


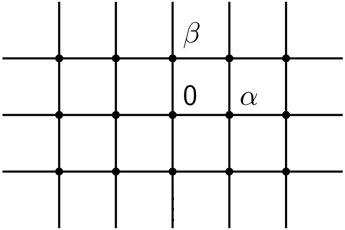

# Proper coloring is an LCL

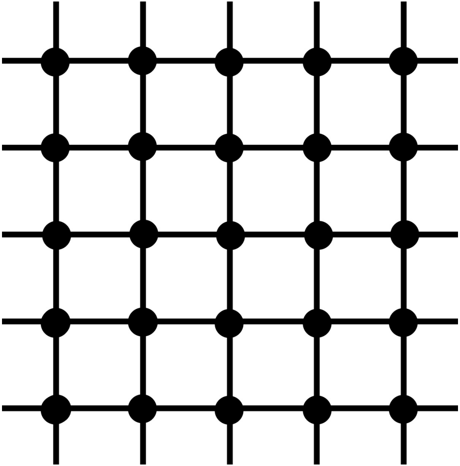

Figure: Proper coloring on $\mathbb{Z}^2$

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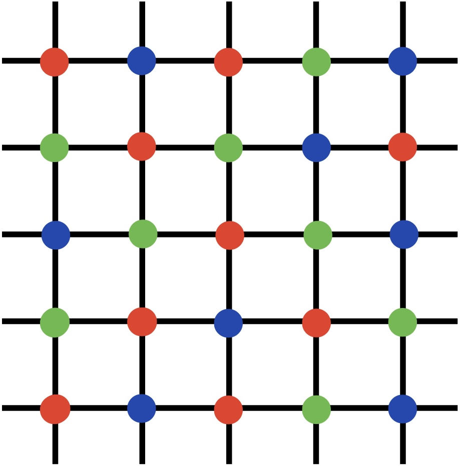

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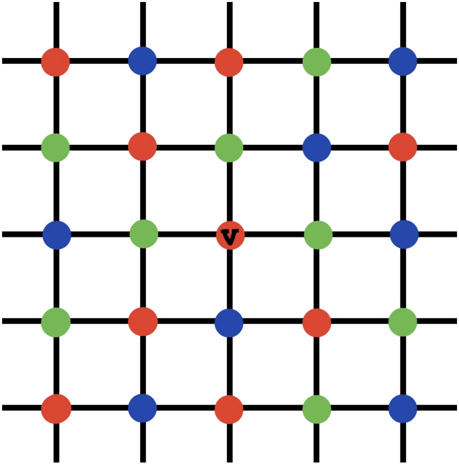


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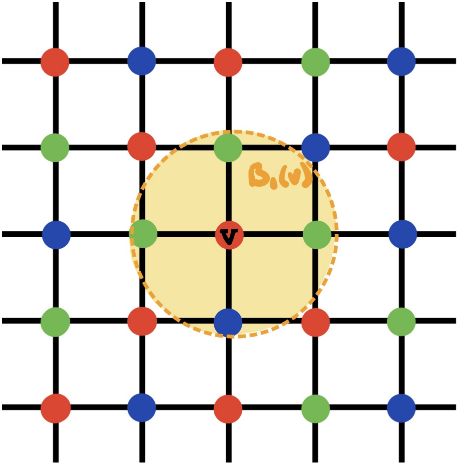


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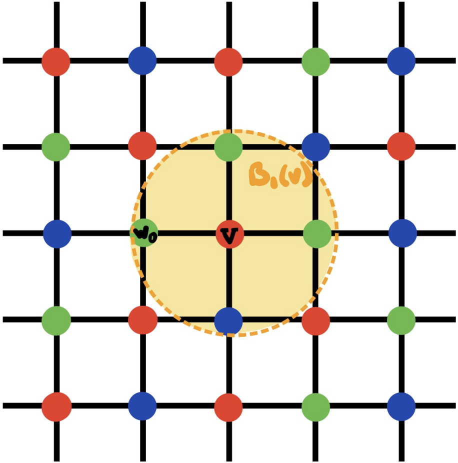


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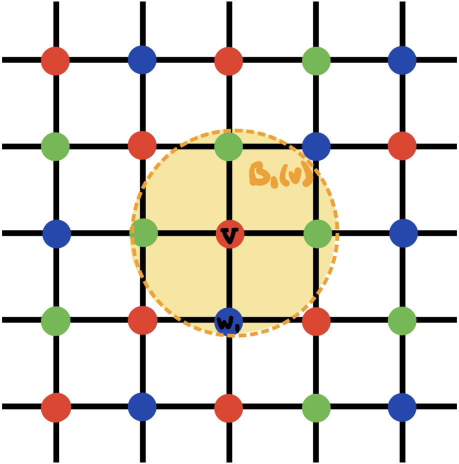


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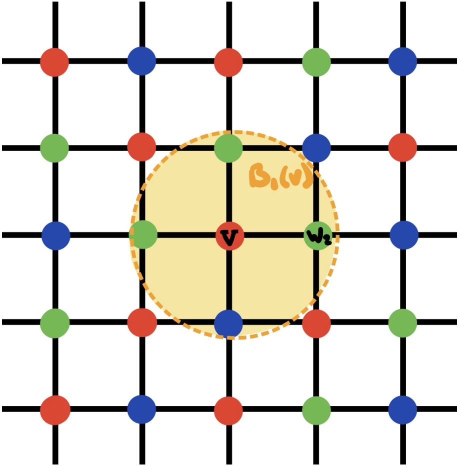


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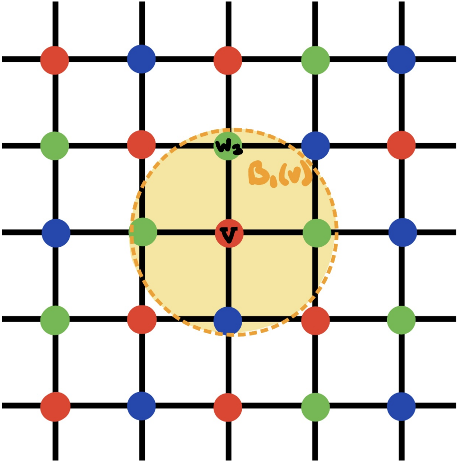


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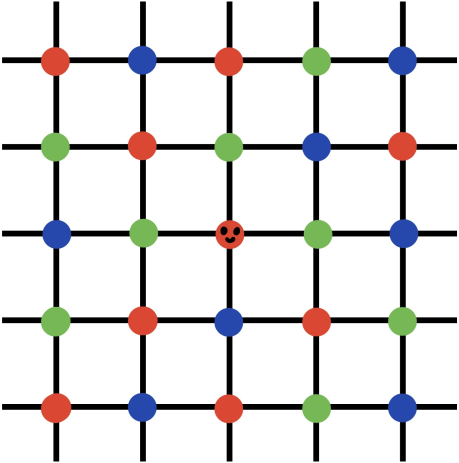

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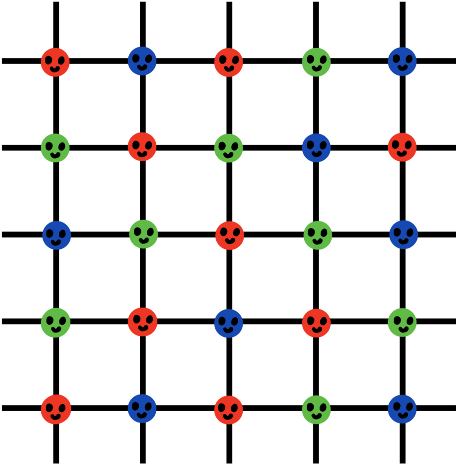

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# Locally Checkable Labellings

## Definition

A **locally checkable labelling problem (LCL)** on a group $\Gamma$ is a triple $\Pi = (W, \Lambda, \mathcal{A})$ where:

- $\Lambda$ is a finite set of **labels**,
- $W \subset \Gamma$ is a finite **window**,
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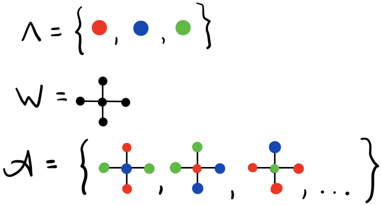

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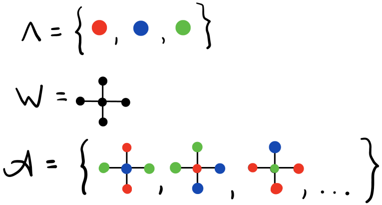


If $G$ is a graph induced by a free action of $\Gamma$ on $X$, we say that a $\Lambda$-labelling of $G$ is a **solution** to $\Pi$ if for every $x \in X$ the function $W \to \Lambda$ given by $\gamma \mapsto c(\gamma \cdot x)$ is in $\mathcal{A}$.

# Locally Checkable Labeling Problems

**Idea**: Label the vertices of $G$ according to some *rule* which can be verified *locally*.

**Examples of LCLs**:

- proper vertex coloring
- proper edge coloring
- matchings
- sinkless orientation
- Wang tiling
- polychromatic coloring

**Nonexamples of LCLs**:

- Hamiltonian cycle
- spanning trees

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# Notions of Definability

## Definition (Descriptive complexity classes)

For an LCL Π, we say:

- $\Pi \in \textbf{CONT}(\Gamma)$ iff every free continuous action $\Gamma \curvearrowright (X, \mathcal{B})$ on a zero-dimensional Polish space admits a **continuous** solution.
- $\Pi \in \textbf{BOREL}(\Gamma)$ iff every free Borel action $\Gamma \curvearrowright (X, \mathcal{B})$ on a standard Borel space admits a **Borel** solution.
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Example: Proper $2n$-coloring is in $\mathrm{MEAS}(\mathbb{F}_n)$ and $\mathrm{BAIRE}(\mathbb{F}_n)$ by Conley–Marks–Tucker-Drob (2016) but not in $\mathrm{BOREL}(\mathbb{F}_n)$ by Marks (2013).

# Notions of Definability (cont.)

The **shift action** is the action $\Gamma \curvearrowright 2^\Gamma$ by $\gamma \cdot x(\delta) = x(\delta \cdot \gamma)$.
We let $\mathcal{F}([0,1], \Gamma)$ denote the free part of the shift action $\Gamma \curvearrowright [0,1]^\Gamma$.

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A measurable $f : \mathcal{F}([0,1],\Gamma) \to \Lambda$ is **finitary** iff for almost all $x \in [0,1]^\Gamma$, there is a finite set $D_x \subseteq \Gamma$ so that if $x|_{D_x} = x'|_{D_x}$ then $f(x) = f(x')$.

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**Definition (Probabilistic complexity classes)**

For an LCL $\Pi$, we say:

- $\Pi \in \mathbf{FIID}(\Gamma)$ iff $\Pi$ admits a measurable solution on $\mathcal{F}([0,1],\Gamma)$.
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# Shift Classes

We may also consider the free part of the shift action in other definability settings.

**Theorem (Seward–Tucker-Drob, Bernshteyn:)**

For any finitely generated group $\Gamma$ and LCL problem $\Pi$ we have:

$$\Pi \in \mathsf{BOREL}(\Gamma) \iff \Pi \text{ has a Borel solution on } \mathcal{F}(\{0,1\},\Gamma).$$

$$\Pi \in \mathsf{CONT}(\Gamma) \iff \Pi \text{ has a continuous solution on } \mathcal{F}(\{0,1\},\Gamma).$$

In the case of Baire measurability, such a result was unknown. We define

**Definition**

Let $\Gamma$ be a finitely generated group. We define

$$\Gamma \in \mathsf{BaireSHIFT}(\Gamma) \iff \Pi \text{ has a Baire measurable solution on } \mathcal{F}(\{0,1\},\Gamma)$$

# Complexity Classes

BOREL, MEAS, BAIRE, FIID, FFIID, . . .

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$\mathrm{BOREL}, \mathrm{MEAS}, \mathrm{BAIRE}, \mathrm{FIID}, \mathrm{FFIID}, \ldots$

**Question:** Are all of these classes distinct? What inclusions can we establish?

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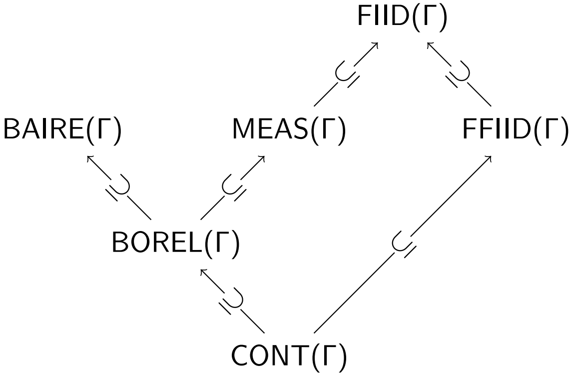


Figure: The trivial inclusions for any Γ.

# Previous Results

Grebík–Rozhoň (2021) have shown:

$$\mathsf{CONT}(\mathbb{Z}) = \mathsf{BOREL}(\mathbb{Z}) = \mathsf{BAIRE}(\mathbb{Z}) = \mathsf{MEAS}(\mathbb{Z}) = \mathsf{FIID}(\mathbb{Z}) = \mathsf{FFIID}(\mathbb{Z})$$

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The following was left open:

- (Grebík–Rozhoň) Is $\mathrm{BOREL}(\mathbb{Z}^n) \subseteq \mathrm{MEAS}(\mathbb{Z}^n)$ strict for $n > 1$?
- Does $\mathrm{MEAS}(\Gamma) \subseteq \mathrm{BAIRE}(\Gamma)$ hold for all $\Gamma$?
- Is $\mathrm{FFIID}(\Gamma) = \mathrm{FIID}(\Gamma)$ for every $\Gamma$?

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# Problems on $\mathbb{Z}^d$

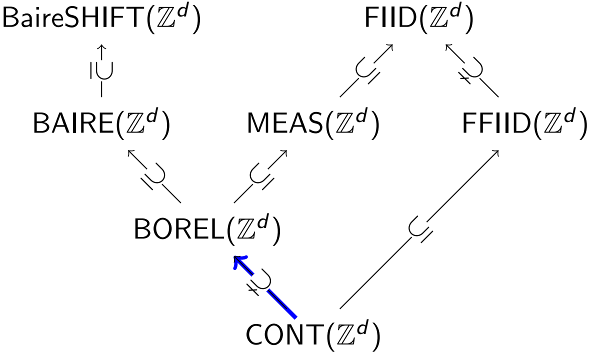


Figure: Complexity classes of LCLs on $\mathbb{Z}^d$, $d \geq 2$.

Blue arrows are strict inclusions. $\subsetneq$

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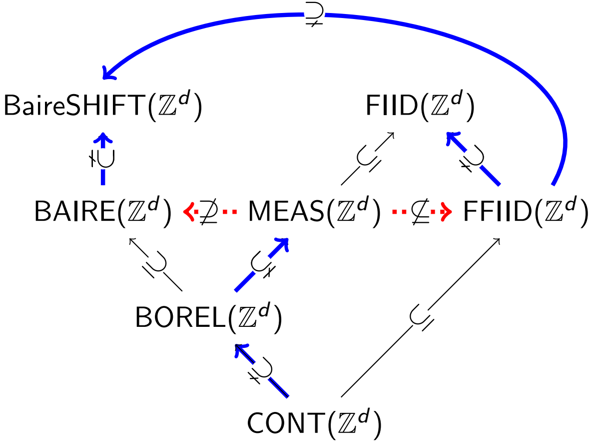


Figure: Complexity classes of LCLs on $\mathbb{Z}^d$, $d \geq 2$.

Blue arrows are strict inclusions. $\subsetneq$
Red dotted arrows are noninclusion. $\not\subseteq$

# Our Results

**Theorem (B.–Bernshteyn–Lyons–Weilacher)**

For $d \geq 2$, there is an LCL $\Pi$ on $\mathbb{Z}^d$ so that:

- $\Pi \in \mathrm{MEAS}(\mathbb{Z}^d)$,
- $\Pi \notin \mathrm{BAIRE}(\mathbb{Z}^d)$,
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This is the first example of a group $\Gamma$ with $\mathrm{MEAS}(\Gamma) \not\subseteq \mathrm{BAIRE}(\Gamma)$ and the first group with $\mathrm{FFIID}(\Gamma) \neq \mathrm{FIID}(\Gamma)$. This is also the first group $\Gamma$ with $\mathrm{BaireSHIFT}(\Gamma) \neq \mathrm{BAIRE}(\Gamma)$.

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# Toast

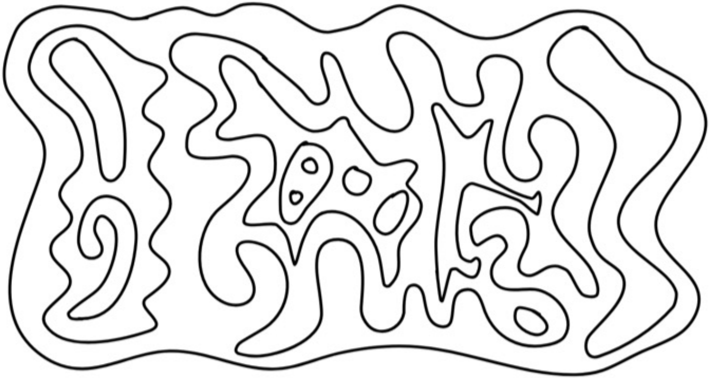

**Figure 1.** A few pieces of a toast.

# Toast

**Definition**

Let $\mathbb{Z}^n \curvearrowright X$ be a free Borel action inducing a graph $G$. We say that a collection of finite sets $\mathcal{T} \subseteq [X]^{<\omega}$ with $\bigcup \mathcal{T} = X$ is a $q$-**toast** if the following two conditions hold for all $K, L \in \mathcal{T}$,

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Note: Borel graphs are hyperfinite iff they admit a 0-toast.

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- either $K \cap L = \varnothing$, $K \subseteq L$, or $L \subseteq K$,
- we have $d(\partial K, \partial L) \geq q$ in the graph metric.

Note: Borel graphs are hyperfinite iff they admit a 0-toast.

**Theorem (Gao–Jackson–Krohne–Seward, 2014-2024):**

Borel graphs induced by free actions of $\mathbb{Z}^d$ on a standard Borel space admit a Borel $q$-toast for any $q \in \mathbb{N}$.

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**Theorem (Gao–Jackson–Krohne–Seward, 2014-2024):**

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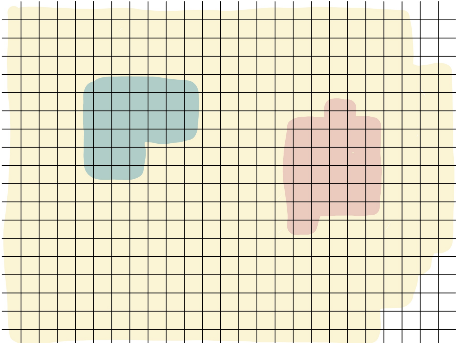

Figure: Proper coloring on $\mathbb{Z}^2$

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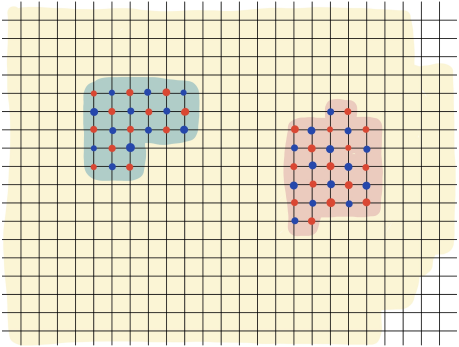

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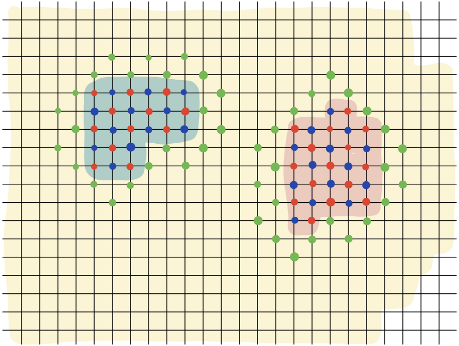

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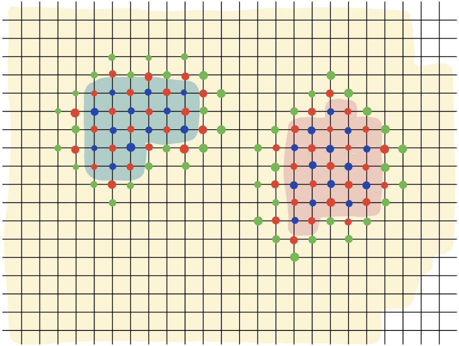

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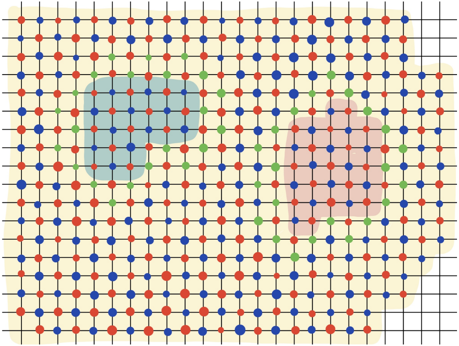

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# Rectangular Toast

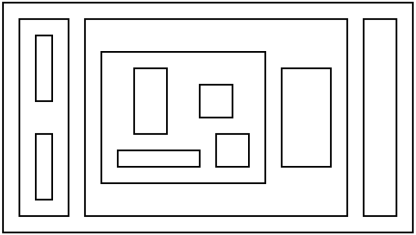

**Figure 3.** A rectangular toast for $\mathbb{Z}^2$.

## Definition

A **rectangular** $q$-toast is a $q$-toast whose pieces are all rectangles.

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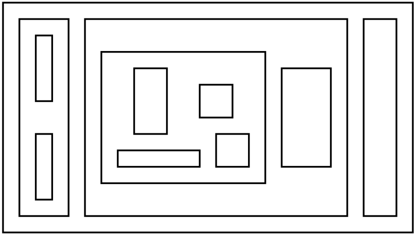

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## Definition

A **rectangular** $q$-toast is a $q$-toast whose pieces are all rectangles.

## Theorem (folklore):

Free Borel actions of $\mathbb{Z}^d$ on a standard probability space admit rectangular $q$-toast on a conull set.

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What if we try to encode rectangular toast as an LCL?

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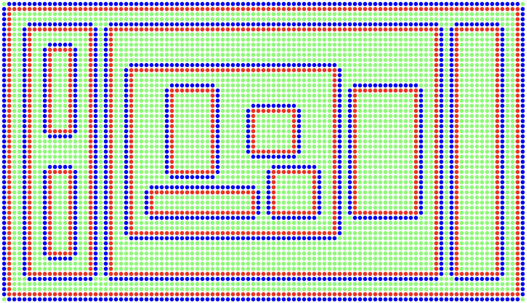

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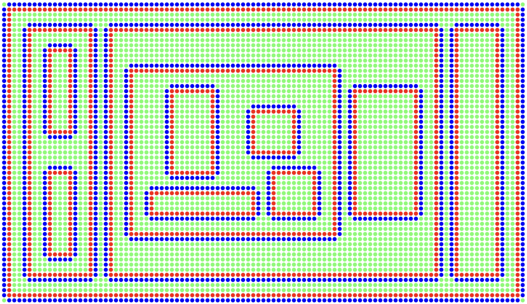
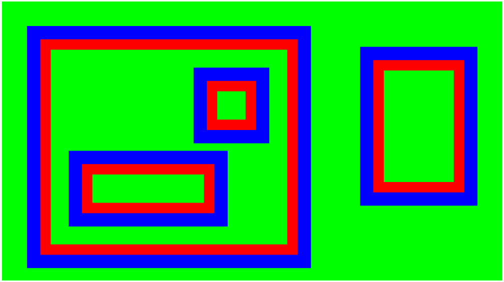
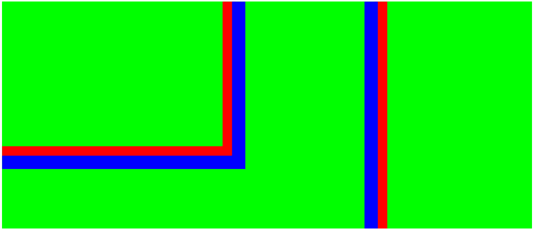


Consider the LCL whose solutions 3-color $\mathbb{Z}^d$ so it *locally* looks like the picture above.

# Issues with this

These diagrams locally look the same.





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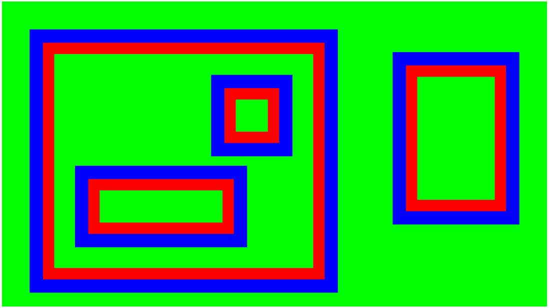
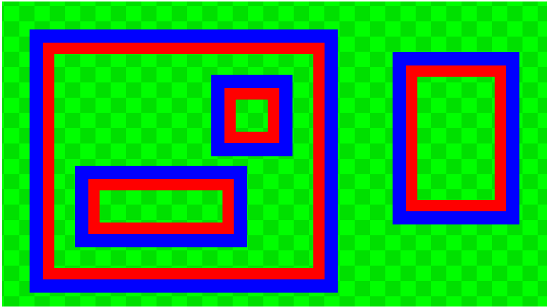


# The Fix

Require the green regions to be 2-colored. This is our new LCL CRT.



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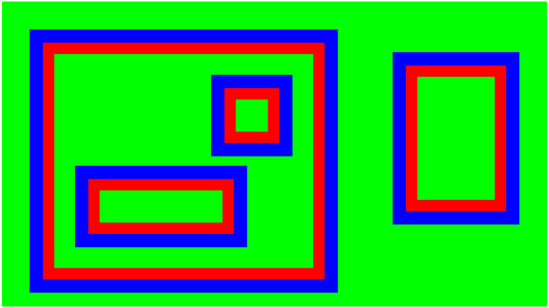

We then have CRT $\in$ MEAS($\mathbb{Z}^d$) by the existence of a measurable rectangular toast. So, in particular, we have CRT $\in$ FIID($\mathbb{Z}^d$).

# CRT has no FFIID Solution

**Theorem (B.–Bernshteyn–Weilacher–Lyons):**

CRT does not always admit a FFIID solution.

*Proof.*

- Let $\mathbb{Z}^d \curvearrowright [0,1]^{\mathbb{Z}^d}$ be the shift action.
  Assume for contradiction $f : X \to \{\text{RED}, \text{BLUE}, \text{GREEN0}, \text{GREEN1}\}$ is an FFIID solution to CRT. Let $\mathcal{T}$ be the corresponding rectangular pre-toast encoded by $f$.

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- We want to show that with probability 1, a chosen point is in only finitely many rectangles. This would give us that the infinite background component admits a FFIID 2-coloring, which would be a contradiction.

$\square$

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- For a fixed $k_1, k_2$, the probability a point $x$ is in a $k_1 \times k_2$ rectangle is the sum over all placements of $x$ in the rectangle of the probability it is in that placement.

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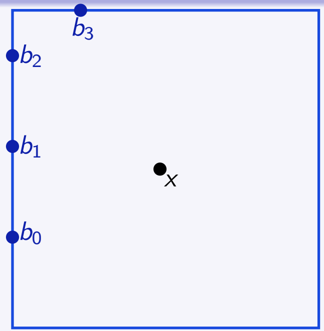


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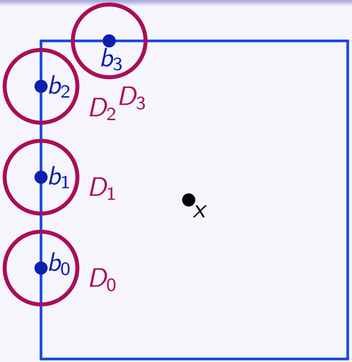


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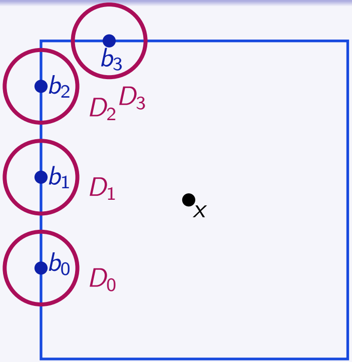


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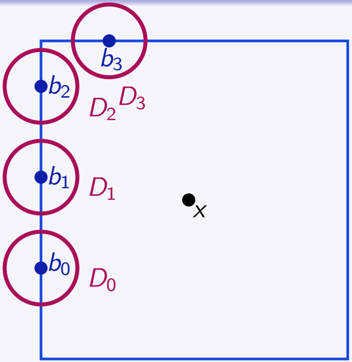
- This is summable over all $k_1$ and $k_2$, thus proving the claim.




□

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# Open Questions

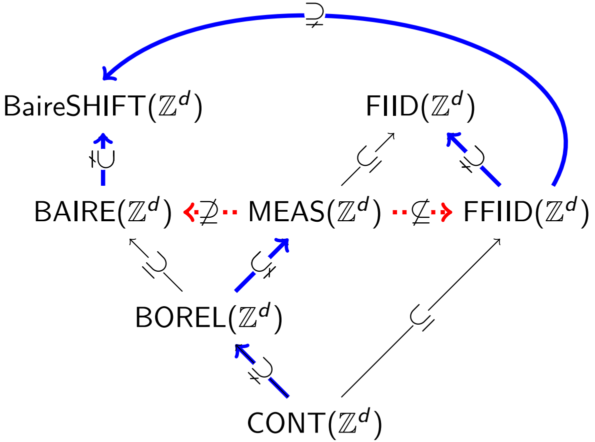


Figure: Complexity classes of LCLs on $\mathbb{Z}^d$, $d \geq 2$.

Some arrows are still missing.

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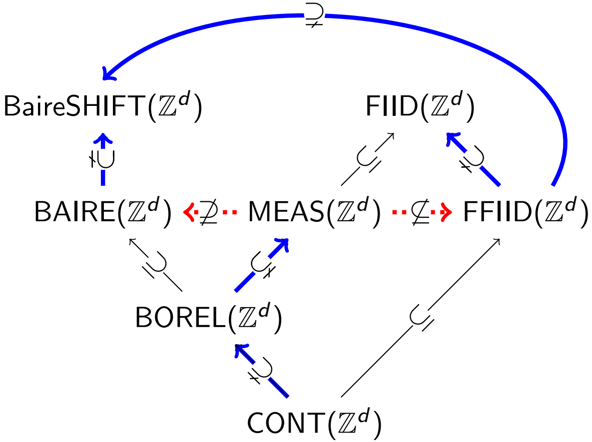


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Thanks!