G_δ circle squaring

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Tarski's circle squaring problem

Question (Tarski's circle squaring problem)

Given a disk and a square in the plane with the same area, are they equidecomposible using isometries?

Let a be an action of a group Γ on a space X.

Definition

Sets $A, B \subseteq X$ are a-equidecomposible if there are a partition $A_1, \ldots A_n$ of A and $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\gamma_1 \cdot A_1, \ldots, \gamma_n \cdot A_n$ is a partition of B.

Solutions

1. (1990) Laczkovich: positive answer to Tarski's circle squaring using translations.
2. (1992) Laczkovich: a general criterion for when two sets of the same positive Lebesgue measure are equidecomposible.
3. (2016) Grabowksi, Máthé and Pikhurko: Lebesgue measurable or Baire measurable pieces in Laczkovich's 1992 theorem.
4. (2017) Marks-U: Borel version of Laczkovich's 1992 theorem.
5. (2022) Máthé-Noel-Pikhurko: Lower Borel complexity and "small boundary" of the pieces.

G_δ circle squaring

Theorem (U-Varadarajan-Weilacher)

Let $A, B \subseteq \mathbb{R}^k$ be bounded sets with $\lambda(A) = \lambda(B) > 0$ and dimension of their boundary less than k. Then A and B are equidecomposable by translations with pieces that are countable unions of Boolean combinations of translates of A, B, and open sets.

In particular, if A, B are a closed disk and square of the same area, then they are equidecomposible with pieces that are both F_σ and G_δ, that is Δ^0_2.

For the closed disk and square, MU gives finite Boolean combinations of Σ^0_4 sets and MNP gives finite Boolean combinations of Σ^0_2 sets.

An outline of a flow based proof of circle squaring

We shift our attention to the Torus $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$.

1. Generate and action of $\mathbb{Z}^d$ on $\mathbb{T}^k$ by translations arranging sufficiently strong "discrepancy estimates".
2. Use the discrepancy estimates to to produce a bounded "flow" ϕ from A to be B. This can be thought of as a real-valued equidecomposition of 1_A and 1_B. ϕ is a uniform limit of its approximations.
3. Convert ϕ to take integer values. This uses a flow rounding algorithm along the layers of a "toast". Most of the Borel complexity is here. Our improvement comes from simpler toast construction.
4. Convert the integer valued flow to an equidecomposition. This is relatively simple.

The flow

A flow in an action is a kind of real valued equidecomposition of the sets A and B via the generators. They should be thought of as coming from averaging procedures.
Let $f_n : \mathbb{T}^k \to \mathbb{R}$ be given by

$$\frac{1}{2^{nd}} \sum_{0 \leq \bar{k} < 2^n} (1_A - 1_B)(\bar{k} \cdot x)$$

Note that the bound in the sum is pointwise. "Discrepancy estimates" refers to bounds on $f_n(x)$ which are independent of x. Moreover, there are flows ϕ_n (which are functions on the edges) that implement the transition from f_n to f_{n+1}.
The final flow is $\sum_{n=0}^{\infty} \phi_n$. The convergence uses the discrepancy estimates.

Rounding

There is an abstract integer flow theorem that will convert a flow to integer values. The instance of this that applies to our graph uses the axiom of choice, so cannot produce anything Borel. Instead we have:

Lemma (Marks-U)

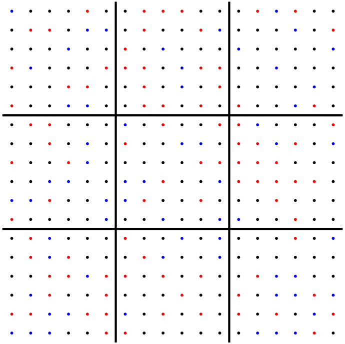
Suppose $d \geq 2$, a is a free Borel action of $\mathbb{Z}^d$ on X, and G is a Schreier graph of a. Then if $f : X \to \mathbb{Z}$ is a Borel function and ϕ is a Borel f-flow for G, then there is an integral Borel f-flow ψ such that $\|\phi - \psi\|_\infty \leq 3^d$.

The lemma is the first use of toast to round a flow to integer values. The complexity of the final integer flow is determined by the flow and Borel complexity of the toast involved.

A refinement of this lemma due to Máthé-Noel-Pikhurko applies a similar rounding procedure to the approximate flows $(\sum_{n=0}^{N} \phi_n)$ to drop the complexity of the resulting integral flow.

Integer-valued flow to an equidecomposition

The equidecomposition can be viewed as a bijection between A and B (red and blue points) with a bound on the distance that a red point moves in a Schreier graph of the action of a.



The red points are mapped between squares when there is a positive flow between those squares. The number of points mapped is based on the amount of flow. The flow condition ensures that there are the same number of remaining red and blue points in each square.

Local constructions and finite Boolean combinations

Let a be an action of a group Γ on a space X and $\mathcal{A}$ be a family of subsets of X. We define $\mathrm{Bool}_a(\mathcal{A})$ to be the set of all finite Boolean combinations of sets $\gamma \cdot A$ for $\gamma \in \Gamma$ and $A \in \mathcal{A}$.

When G is a Schreier graph of the action a, sets in $\mathrm{Bool}_a(\mathcal{A})$ can be thought of those that are "G-local functions" of sets in $\mathcal{A}$.

In particular the previous argument converting the integer flow ψ to an equidecomposition, each set A_i in the equidecomposition is in $\mathrm{Bool}_a(\{\psi^{-1}(n) \mid n \leq \|\psi\|_\infty\} \cup \{\text{ Basic Open sets}\} \cup \{A, B\})$.

The class $\Delta_\Gamma(\mathcal{A})$

Let a be an action of Γ on X.

We define $\Sigma_a(\mathcal{A})$ to the collection of countable unions of elements of $\Gamma \cdot \mathcal{A}$ and similarly for $\Pi_a(\mathcal{A})$ for intersections.

We define $\Delta_a(\mathcal{A})$ to be sets that are in both $\Sigma_a(\mathcal{A})$ and $\Pi_a(\mathcal{A})$.

The following fact is easy but important

Proposition

The class $\Delta_a(\mathcal{A})$ is an a-invariant algebra and $\Delta_a(\Delta_a(\mathcal{A})) = \Delta_a(\mathcal{A})$.

We sometimes drop the subscript a in the context where we can ignore the action.

Toast

Let G be a Borel graph on a space X.

Definition (Gao-Jackson-Krohne-Seward)

A *q-toast* is a sequence $(T_n)_{n\in\mathbb{N}}$ of subsets of X (called *layers*) such that

1. $\bigcup_n T_n = X$.
2. For each n, the components of $G^{\leq q} \upharpoonright T_n$ (called *pieces*) have uniformly bounded diameter.
3. For each $n < m$, $\mathrm{dist}(T_n, \partial_i T_m) \geq q$.

Here $G^{\leq q}$ is the graph on X where two vertices are adjacent if they have distance at most q in G.

Toast 2

Theorem (Gao-Jackson-Seward-Krohne)

Let a be a free continuous action of $\mathbb{Z}^d$ on a 0-dimensional Polish space X with Schreier graph G. For any q, G admits a q-toast with Δ^0_2-layers.

More generally, we show

Theorem (Varadarajan-Weilacher-U)

Let a be an action of Γ on X and G be a Schreier graph of a. If a admits "witnesses to asymptotic dimension" from a family of sets $\mathcal{A}$, then for every q, G has a q-toast such that each layer is in $\Delta(\Sigma(\mathcal{A}))$

Borel Asymptotic dimension

Introduced by Conley, Jackson, Marks, Seward, Tucker-Drob. Let G be a Schreier graph of an action of a finitely generated group Γ on X.

Definition

Let $d, r \in \mathbb{N}$. An *asymptotic dimension d witness for G at scale r* is a cover of X by sets $U^0, \ldots, U^d$ such that for each i, the connected components of $G^{\leq r} \upharpoonright U_i$ have uniformly bounded diameter.
The *asymptotic dimension* of G is the least d such that G admits such a witness at every scale r, or ∞ if there is no such d. We write $d = \mathrm{asdim}(G)$.

It is natural to restrict the sets U^i in the covers above to some class of subsets of X. For instance, every free continuous action of $\mathbb{Z}^d$ on a 0-dimensional Polish space has asymptotic dimension d using clopen sets.

Rainbow toast

We start by taking witnesses U_n^i for $i \leq d$ to be asymptotic dimension witnesses for $n \in \mathbb{N}$ where the scale increases with n.

$$\begin{matrix} U_0^0 & U_1^0 & \cdots & U_d^0 \\ U_0^1 & U_1^1 & \cdots & U_d^1 \\ U_0^2 & U_1^2 & \cdots & U_d^2 \\ \vdots & \vdots & \ddots & \vdots \end{matrix} \qquad \begin{matrix} T_0^0 & T_1^0 & \cdots & T_d^0 \\ T_0^1 & T_1^1 & \cdots & T_d^1 \\ T_0^2 & T_1^2 & \cdots & T_d^2 \\ \vdots & \vdots & \ddots & \vdots \end{matrix}$$

The sets T_n^i are the result of disjointifying the boundaries of the U_n^i sets. As a result each of the columns will satisfy parts (2) and (3) in the definition of toast.

These $d + 1$ toasts can be made into a toast by choosing for each x an index i such that T_n^i covers x infinitely often. Unfortunately this construction cannot give the optimal complexity.

Bounded geometry decomposition

Similar to "weakly orthogonal decomposition" of Gao-Jackson-Krohne-Seward we define:

Definition

We say that a sequence $(X_n)_{n\in\mathbb{N}}$ of subsets of X is a *bounded geometry decomposition* (BGD) on G with constants P and Q if, letting $X_n^\infty = \bigcup_{k\geq n} X_k$, the following hold:

1. $\bigcap_n X_n^\infty = \emptyset$.
2. For each n, there is a uniform bound D_n on the diameters of connected components of $G \restriction (X \setminus X_n^\infty)$.
3. For any component R of $G \restriction (X \setminus X_n^\infty)$, the set $\{k \geq n \mid \mathrm{dist}(R, X_k) \leq Q\}$ has cardinality at most P.

Low complexity on the torus

Lemma

In the context of the sequences T_n^i for $i \leq d$ and $n \in \mathbb{N}$, for any Q there is a sequence of indices (r_n) such that the sets $X_n = \bigcup_i \partial T_{r_n}^i$ form a BGD with constants $d+2$ and Q.

Adapting the construction above to the action of translations on the torus, the sets X_n from the lemma will be in $\Sigma(\mathrm{Bool}_a(\mathcal{B}))$ where $\mathcal{B}$ is some basis for the torus.

We observe in the proof of the previous lemma we can selectively take the interior of the sets X_n and use a faster sequence (r_n) to get a BGD with open sets.

Our action on the torus is almost 0-dimensional

The proof above makes use of the following observation:

If a is an action of translations on $\mathbb{T}^k$ that are chosen uniformly at random and $B \subseteq \mathbb{T}^k$ is an axis parallel box with rational corners, then the topological boundary of B meets each orbit of a at most once. This extends to Boolean combinations in the obvious way.

More generally, we define a class of group actions which are "pseudo-0-dimensional", which captures this feature of the torus and our arguments work for these spaces.

Toast

Adapting work of Gao-Jackson-Krohne-Seward, we have:

Theorem

Let $(X_n)_{n\in\mathbb{N}}$ be a BGD on G with constants P and Q, and choose $q \in \mathbb{N}$ so that $q(P+1) \leq Q$. Then, there is a q-toast $(T_n)_{n\in\mathbb{N}}$ on G such that for each $n \in \mathbb{N}$, $T_n \in \Delta_\Gamma\left(\{X_n, X_n^\infty \mid n \in \mathbb{N}\}\right)$.

From our construction of a bounded geometry decomposition of $\mathbb{T}^k$ with open sets, it follows that we have a toast with layers in $\Delta_a(\{\text{Basic open sets}\}) = \Delta^0_2$.

Thanks!