Linings, Treeings, and the Continuous Automorphism Problem

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March 11, 2026
Caltech Logic Seminar

Parts are joint with C. Olsen and parts with M. Kang.

The subject of investigation concerns descriptive combinatorics and continuous dynamics.

A basic question is what kinds of “structures” can be put on the equivalence classes of a countable Borel equivalence relation (cber) is a definable manner, specifically a continuous or Borel manner?

This subject was initiated in the seminal Kechris-Solecki-Todorcevic paper.

The main focus here concerns continuous types of structurings on the classes. There are several ways to make these notions precise.

Recall the Feldman-Moore theorem:

Theorem (Feldman-Moore)

Every cber is the orbit equivalence relation of the Borel action of a countable group.

Of particular importance is the (left) action of a countable groups G on Z^G, usually $Z = 2$ or ω, or ω^ω.

$$g \cdot x(h) = x(g^{-1}h)$$

Fact

Every continuous action of G on a 0-dimensional Polish space continuously embeds into the shift action of G on $(\omega^\omega)^G$ and into the shift action of G on $2^{G\times\omega}$.

A case of particular interest in continuous dynamics is the shift action of $\mathbb{Z}^n$ on $2^{\mathbb{Z}^n}$, $E(\mathbb{Z}^n)$, and its free part which we denote $F(2^{\mathbb{Z}^n})$.

On the one hand, all of these relations are hyperfinite, and in fact there are continuous embeddings into E_0 (even on the the entire $X = 2^{\mathbb{Z}^n}$).

On the other hand, it is known that there are some "continuous structuring" questions for which the answer depends on the dimension.

- For $n = 1$, the continuous chromatic number of $F(2^{\mathbb{Z}^n})$ is equal to the Borel chromatic number ($= 3$) but for $n > 1$ it is strictly greater (Borel$= 3$, continuous$= 4$).
- For $n \leq 2$, there are continuous "minimal" rectangular partitions of $F(2^{\mathbb{Z}^2})$, but for $n \geq 3$ there are not (Gao, J).

Question

Are there natural continuous structuring questions which separate the $F(2^{\mathbb{Z}^n})$?

One notion of "continuous structuring" is the following.

Definition

Let $R \subseteq X^n$ where $R(x_1, \dots, x_n)$ implies $[x_1] = \dots = [x_n]$. We say R is G-clopen if for all $g_1, \dots, g_n \in G$, $\{x \colon R(g_1 \cdot x, \dots, g_n \cdot x)\}$ is clopen in X.

Usually $X = F(2^G)$.

Line sections

For example we have the following notion of "line section":

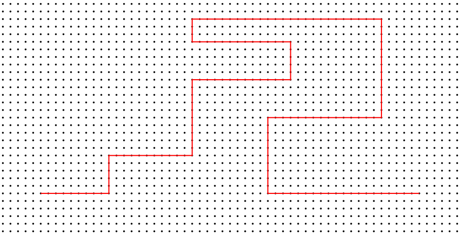
Definition

Let $G = \langle s_1, \dots, s_n \rangle$ and $F(2^G)$ the free part of the shift action. A (Schreier) line-section L is a subgraph of the Schreier graph such that on each equivalence class $[x]$, $L \restriction [x]$ is isomorphic to the graph of $\mathbb{Z}$.

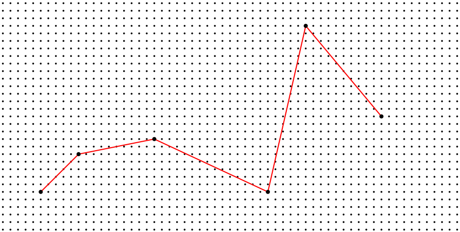
Definition

A (general) line-section L is a subgraph of (X, E) such that on each equivalence class $[x]$, $L \restriction [x]$ is isomorphic to the graph of $\mathbb{Z}$.

In both cases we have the notion of the line-section being G-clopen.



(a) Schreier line-section



(b) general line-section

Tree sections

We similarly define the notion of a Schreier or general tree-section, and the notion of it being G-clopen.

We say a line/tree section is total if its vertex set is all of X.

At the Borel level, a total line-section is just a Borel $\mathbb{Z}$-ordering of each class, that is, it is a Borel witness to hyperfiniteness. So, this is equivalent to have a Borel automorphism generating E.

Continuous automorphism problem

The continuous automorphism problem asks whether there is a continuous $\varphi\colon F(2^G) \to F(2^G)$ such that on each $[x]$, φ induces a total $\mathbb{Z}$-ordering of $[x]$.

Fact

If a line-section is G-clopen then it is continuous as a partial function from X to X.

In particular, we have two notions of a "continuous automorphism" generating E: being G-clopen, and the map being topologically continuous, with the first implying the second.

Again, at the Borel level both are equivalent to being hyperfinite.

At the continuous level there are also other candidates for "continuous hyperfiniteness."

Assume $G \curvearrowright X$ continuous, X is 0-dimensional:

1. $E_G = \bigcup_n E_n$ an increasing union where each E_n is a finite G-clopen equivalence relation.
2. $E_G = \liminf E_n$ where each E_n is a finite G-clopen equivalence relation.
3. There is a continuous one-to-one $f \colon X \to 2^\omega$ witnessing $E \sqsubseteq E_0$.
4. There is a continuous reduction of E to E_0.

Fact

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4)

The non-trivial implication is (2) $\Rightarrow$ (3) [Kang,J].

Theorem (Kang,J)

- *(1) does not hold for any infinite, finitely generated group.*
- *[Conley, Marks, J, Seward, Tucker-Drob] Let $G \curvearrowright X$ a continuous action with X 0-dimensional. If E_G has finite continuous asymptotic dimension, then $E = \liminf E_n$ where the E_n are G-clopen uniformly bounded subequivalence relations. So (2) holds.*
- *If G has an element of infinite order then there does not exist a G-clopen finite subequivalence relation for the full shift on 2^G.*

Thus, (2)$\not\Rightarrow$ (1) and (3)$\not\Rightarrow$ (2).

Returning to the continuous automorphism problem we have the following.

Theorem (Giordano-Matui-Putnam-Skau)

A minimal subflow of $F(2^{\mathbb{Z}^n})$ *is generated by a continuous automorphism.*

Many constructions of hyperaperiodic elements give minimal elements, so hyperaperiodicity is not enough to rule out a generating continuous automorphism.

Recall $x \in 2^G$ is hyperaperiodic if $\overline{[x]} \subseteq F(2^G)$.

Hyperaperiodicity is enough to rule out G-clopen line-sections and other related structurings.

The first result along these lines is:

Theorem (Gao, J, Krohne, Seward)

Let $x \in F(2^{\mathbb{Z}^n})$ be hyperaperiodic. Then there does not exist a G-clopen line section on $\overline{[x]}$ (on each class isomorphic to an action of $\mathbb{Z}$).

More recently this result has been strengthened in the following ways.

These results also show a difference between "open" and "clopen" G-structurings.

Lining and treeing results

We have the notions of k-line sections and $\leq k$ line-sections (on each class there are exactly k (or $\leq k$ respectively) lines (for either Schrerier or general version), and likewise for tree-sections.

In the theorems below we require the tree sections to be either Schreier or of fixed vertex degree.

Theorem (Olsen, J)

There does not exist a G-open k tree-section of $F(2^{\mathbb{Z}^n})$. In particular there is no k line-section.

Theorem (Olsen, J)

There does not exist a G-clopen $\leq k$ tree-section of $F(2^{\mathbb{Z}^n})$. In particular there is no $\leq k$ line-section.

If we say “open” and $\leq k$” then it is possible:

Theorem (Olsen,J)

There is an open ≤ 3 Schreier tree-section of $F(2^{\mathbb{Z}^2})$. There is an open ≤ 4 Schreier line-section of $F(2^{\mathbb{Z}^2})$.

The first line-section result has the following consequence noticed by Kang and J.

Theorem

Let $x \in F(2^{\mathbb{Z}^n})$ be hyperaperiodic. Then there does not exists a continuous automorphism φ generating $E \restriction K = \overline{[x]}$ with the property that on each class $[y] \in K$, $\varphi(y)$ moves y a bounded distance in the Schreier graph.

Remark

G-clopen (general) line-sections must move points a bounded distance in the Schreier graph on the orbit closures of every hyperaperiodic element.

Proof. As x is hyperaperiodic, $K \subseteq F(2^{\mathbb{Z}^n})$ and is compact. We show that $\varphi \upharpoonright K$ is actually G-clopen. Say $\varphi(y)$ is at most distance N from y.

For all $y \in K$, $\varphi(y) = g \cdot y$ iff $\|g\| \leq N$ and $\forall h \in G$ with $\|h\| \leq N$, if $h \neq g$ then $\varphi(y) \neq h \cdot y$.

This shows $\varphi(y) = g \cdot y$ is open, and it is clearly closed, so clopen.

We sketch the proof that there is no $\leq k$ clopen (general) line-section or tree-section.

Recall $x \in 2^G$ is hyperaperiodic iff for all $s \in G \setminus \{e\}$ there is a $T \in G^{<\omega}$ such that

$$\forall g \in G\ \exists t \in T\ x(gt) \neq x(gst)$$

Assume now $G = \mathbb{Z}^2$. Towards a contradiction, assume T is a G-clopen $\leq k$ treeing of $F(2^{\mathbb{Z}^2})$.

Consider basic open sets in $2^{\mathbb{Z}^2}$ of the form $\{x \in 2^{\mathbb{Z}^2} : x \upharpoonright R = p\}$ where $R \subseteq \mathbb{Z}^2$ is a square and $p \in 2^R$. So, we identify basic open sets with squares of 0's and 1's.

Good pairs

Definition

A basic open set M is good if when we let N be the 3×3 tiling by copies of M, then for any point x in M, N determines if $x \in T$.

First assume that a good M exists.

Let $y \in F(2^{\mathbb{Z}^2})$ differ from a tiling by copies of M in a finite set.

Let T_y be the treeing on $[y]$, with components $T_y^0, \dots, T_y^{\ell-1}$.

Let $B \subseteq \mathbb{Z}^2$ be the points of distance $\leq 2|M|$ from the finite set.

Let $z \in T_y$ with $z \notin B \cdot y$. By the goodness of M, there is a copy of N in y that determines that $z \in T$.

Let $z_0 = z$ and define z_{i+1} to be the shift of z_i to the right by $|M|$. So $z_i \in T$ for all i.

We can get a subsequence z_{i_k} all in T_0^y, say. Let p_k be the path in T_0^y connecting z_{i_k} and $z_{i_{k+1}}$. For large enough k, $p = p_k \cap B = \emptyset$.

We can create an arbitrarily long sequence of copies of M spaced by $d(z_{i_{k+1}}, z_{i_k})$. Call these points $w_0, w_1, \dots$. These are connected by paths in T_0^y. The paths connecting w_i and w_{i+1} are all isomorphic.

We choose the length of the sequence to be large compared to the length of p.

We now repeat the argument using a sequence of copies of M placed vertically.

We get a sequence of points $u_0, u_1, \dots$ all of which are in the same relative placement in their copies of M. The paths in T^y connecting u_i and u_{i+1} are all isomorphic to a path q.

We can then take $u_0 = w_\ell$ as we may take ℓ large enough comapred to $|q|$.

We can then complete the rectangle of copies of M which gives a cycle in T^y_0, a contradiction.

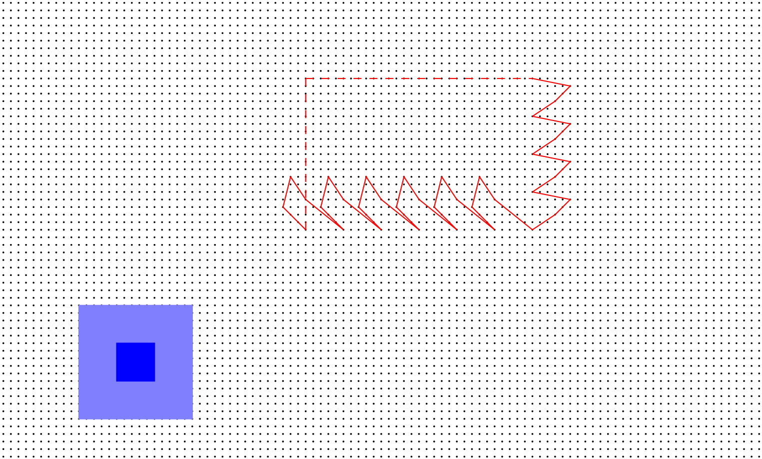


Figure: constructing a cycle

Lemma

There is a good pair (M, N).

Proof: Let $A_0 = A$ be an $n_0 \times n_0$ grid, and $\overline{A_0}$ the flip of A (n_0 and A can be varied).

Let A_{n+1} be the 7×7 tiling by copies of $\overline{A}_n$ except the central copy is A_n.

Let $x \cup A_n$. Note that for each n, x is tiled by copies of A_n and $\overline{A}_n$. x is easily hyperaperiodic and minimal.

Let $K = \overline{[x]} \subseteq F(2^{\mathbb{Z}^2})$.

By compactness, there is an n such that for all $y \in K$, $y \upharpoonright [-n, n]^2$ determines if $y \in T$.

At stage n in the construction of x we have a good pair.

Completely non-minimal

We introduce the notion of a completely non-minimal element.

Recall that for a continuous action $G \curvearrowright X$, $x \in X$ is minimal if $\overline{[x]}$ is a minimal closed invariant set. This is equivalent to saying every open set U occurs syndetically.

Definition

$x \in F(2^{\mathbb{Z}^2})$ is completely non-minimal if there is an $n \in \omega$ such that for every $n \times n$ grid N which appears in x, there are arbitrarily large $m \times m$ grids in x which do not contain a copy of N.

Fact (J, Kang)

There exist hyperaperiodic completely non-minimal $x \in F(2^{\mathbb{Z}^2})$.

sketch: We construct two particular hyperaperiodic elements, y, z.

Let A be the 4×4 grid $(-1)^{i+j}$, and B the 4×4 grid of 0's.

From $A, \bar{A}$ (the flip of A) build the hyperaperiodic element y, and build z from B and $\bar{B}$.

Let x be built from alternating vertcal strips of increasing widths of portions of y and z. Easily x is hyperaperiodic.

The complete non-minimality follows from:

Fact

For every 5×5 grid A, either A does not occur in y or A does not occur in Z.

Almost invariant regions

We state another result relating to the continuous automorphism problem.

Theorem (almost invariant regions)

Let $\varphi\colon F(2^{\mathbb{Z}^2}) \to F(2^{\mathbb{Z}^2})$ *continuous, one-to-one, and such that* $\varphi(x) E x$*. Then there is a* $d > 0$ *such that for every* $n \in \omega$ *there is an* $x \in F(2^{\mathbb{Z}^2})$ *and an* $n \times n$ *rectangle* R *in* $[x]$ *such that* $\varphi(R)$ *is contained in the* d*-expansion of* R*.*

Proof: Let (M, N) be a good pair for φ in the sense that for every $x \in F(2^{\mathbb{Z}^2})$ in a copy of M in a (M, N) pair, $\varphi(x) \upharpoonright G$ is determined, where G is the 3×3 grid. We may assume M is a checkerboard pattern. Say M is $d \times d$.

Let $y \in F(2^{\mathbb{Z}^2})$ be tiled by copies of M except for a single copy of the all 0's $d \times d$ grid.

As φ is one-to-one, there is a copy of (M, N) in y such that for all z in the copy of M, $\varphi(z) \restriction G$ is a checkerboard pattern.

Thus, the good pair (M, N) determines that all points z in an (M, N) neighborhood see a 3×3 checkerboard pattern around $\varphi(z)$.

For any n, let $u_n \in F(2^{\mathbb{Z}^2})$ be all 0's except for an $n \times n$ tiling of copies of M.

Let $R_n \subseteq [u_n]$ be the inner $(n-1) \times (n-1)$ tiling of copies of M. Then $\varphi(R_n) \subseteq R$.

Summary

To summarize, for the automorphism problem we have:

Theorem (unbounded distances)

Let φ be a continuous automorphism generating $F(2^{\mathbb{Z}^2})$. Then there does not exist an N such that $d(\varphi(x), x) \leq N$ for all x.

Theorem (almost invariant regions)

Let φ be a continuous automorphism of $F(2^{\mathbb{Z}^2})$. Then there is a $d > 0$ and an $x \in F(2^{\mathbb{Z}^2})$ such that for arbitrarily large N we have a rectangle $R \subseteq [x]$ with $\varphi(R) \subseteq R_d$, the d-expansion of R.