Manifold classification from the descriptive viewpoint

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Caltech Logic Seminar

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This talk will, inevitably, overlap a little bit with that one. Within it, though, I'll also at least *try* to reach a part of our paper that neither of us has presented in any depth; these are its last sections, on the classification, up to isometry, of hyperbolic n-manifolds with finitely generated fundamental group.

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- it furnished a pretext for learning things that I did genuinely wish and like to learn, and in the process
- it gave me things to keep in touch with my friend and mathematical brother, the actual descriptive set theorist Iian Smythe, about.

for example (I)

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ON THE CLASSIFICATION OF NONCOMPACT SURFACES

BY
IAN RICHARDS(1)

1. **Introduction.** In this paper we describe a complete topological classification of noncompact triangulable surfaces, and give a concrete model for an arbitrary surface, similar to the classical "normal form" for compact surfaces.

This classification of arbitrary surfaces depends on the well-known classification theorem for compact surfaces, and on the idea, frequently used in the theory of Riemann surfaces, of the "ideal boundary" of a surface. The ideal boundary is a totally disconnected, compact, separable space. For our purposes, we distinguish two nested closed subsets of this space, corresponding to portions of the surface which are of "infinite genus" and "infinitely nonorientable" respectively; thus our "ideal boundary" is really a nested triple of spaces.

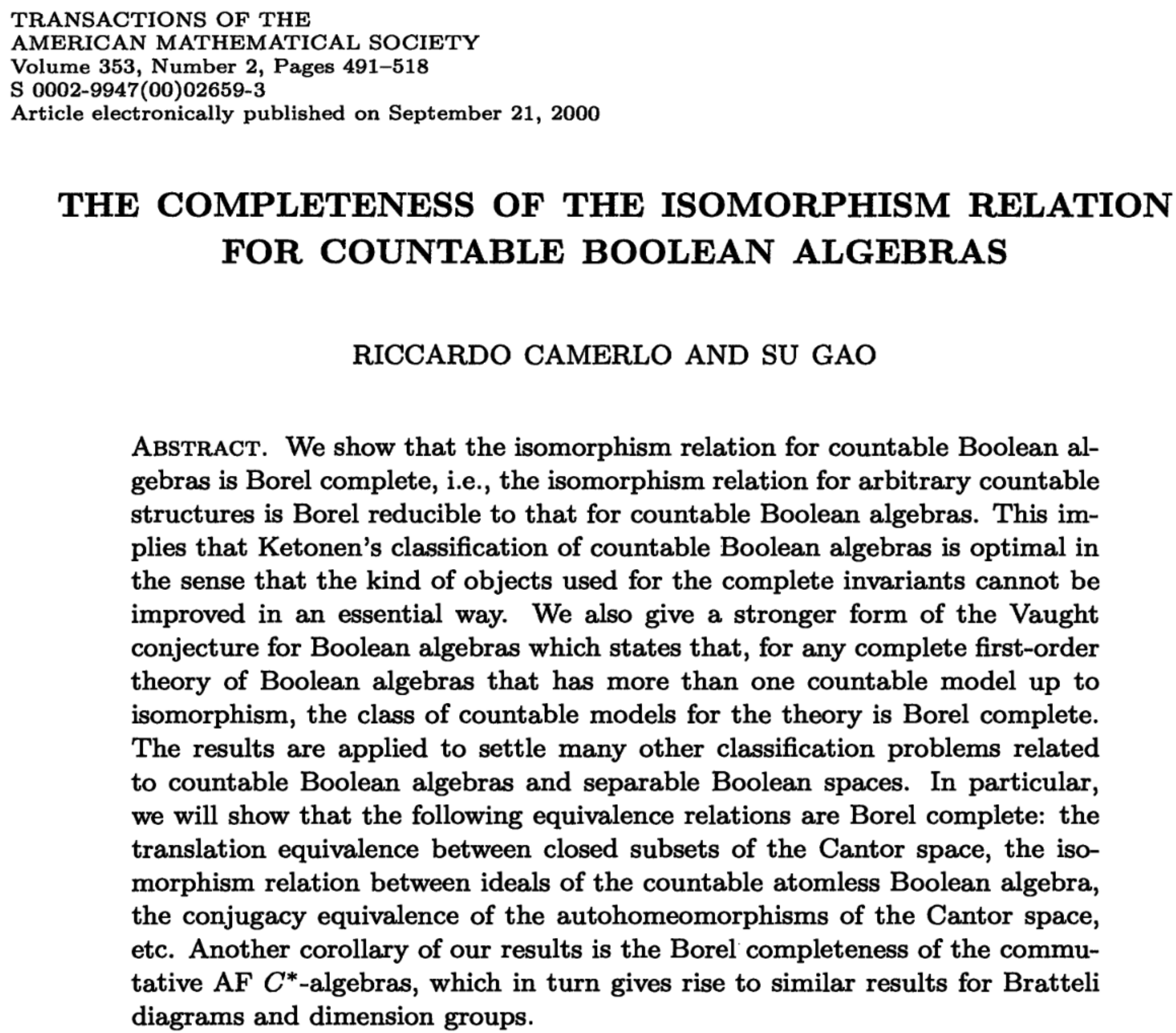
Our first result is that, with certain fairly obvious qualifications, two surfaces are homeomorphic if and only if their ideal boundaries are topologically equivalent. This was originally discovered by Kerékjártó (see Kerékjártó [5, Chapter 5]). Kerékjártó's proof seems to contain certain gaps, so we have included an outline of a complete proof. (See in particular the remark following Proposition 3 in §3.)

for example (I)

4. **Kerékjártó's Theorem.**

THEOREM 1. *Let S and S' be two separable surfaces of the same genus and orientability class. Then S and S' are homeomorphic if and only if their ideal boundaries* (*considered as triples of spaces*) *are topologically equivalent.*

for example (I)


TRANSACTIONS OF THE
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S 0002-9947(00)02659-3
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THE COMPLETENESS OF THE ISOMORPHISM RELATION FOR COUNTABLE BOOLEAN ALGEBRAS

RICCARDO CAMERLO AND SU GAO

ABSTRACT. We show that the isomorphism relation for countable Boolean algebras is Borel complete, i.e., the isomorphism relation for arbitrary countable structures is Borel reducible to that for countable Boolean algebras. This implies that Ketonen's classification of countable Boolean algebras is optimal in the sense that the kind of objects used for the complete invariants cannot be improved in an essential way. We also give a stronger form of the Vaught conjecture for Boolean algebras which states that, for any complete first-order theory of Boolean algebras that has more than one countable model up to isomorphism, the class of countable models for the theory is Borel complete. The results are applied to settle many other classification problems related to countable Boolean algebras and separable Boolean spaces. In particular, we will show that the following equivalence relations are Borel complete: the translation equivalence between closed subsets of the Cantor space, the isomorphism relation between ideals of the countable atomless Boolean algebra, the conjugacy equivalence of the autohomeomorphisms of the Cantor space, etc. Another corollary of our results is the Borel completeness of the commutative AF C^*-algebras, which in turn gives rise to similar results for Bratteli diagrams and dimension groups.

for example (I)

Theorem. *The class of all countable Boolean algebras is Borel complete.*

This has a topological counterpart.

Theorem. *The homeomorphism relation between separable Boolean spaces, i.e., zero-dimensional compact metrizable spaces, is Borel complete.*

for example (II)

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Annals of Mathematics **176** (2012), 1–149
http://dx.doi.org/10.4007/annals.2012.176.1.1

The classification of Kleinian surface groups, II: The Ending Lamination Conjecture

By Jeffrey F. Brock, Richard D. Canary, and Yair N. Minsky

Abstract

Thurston's Ending Lamination Conjecture states that a hyperbolic 3-manifold N with finitely generated fundamental group is uniquely determined by its topological type and its end invariants. In this paper we prove this conjecture for Kleinian surface groups; the general case when N has incompressible ends relative to its cusps follows readily. The main ingredient is a uniformly bilipschitz model for the quotient of $\mathbb{H}^3$ by a Kleinian surface group.

for example (II)

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Thurston's scheme proposes *end invariants* that encode the asymptotic geometry of the ends of the manifold, generalizing the role the Riemann surfaces at infinity play in the geometrically finite case. More precisely, the following conjecture appears in [76].

ENDING LAMINATION CONJECTURE. *A hyperbolic* 3*-manifold with finitely generated fundamental group is uniquely determined by its topological type and its end invariants.*

This paper is the second in a series of three which will establish the Ending Lamination Conjecture for all topologically tame hyperbolic 3-manifolds. For

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We'd admired this work since graduate school together; arguably our main mathematical ambition in these conversations was simply to relate to it, to extend it, in some nontrivial way.

example III

example III

ILLINOIS JOURNAL OF MATHEMATICS
Volume 44, Number 1, Spring 2000

THE COMPLEXITY OF THE CLASSIFICATION OF RIEMANN SURFACES AND COMPLEX MANIFOLDS

G. HJORTH AND A.S. KECHRIS

ABSTRACT. In answer to a question by Becker, Rubel, and Henson, we show that countable subsets of $\mathbb{C}$ can be used as complete invariants for Riemann surfaces considered up to conformal equivalence, and that this equivalence relation is itself Borel in a natural Borel structure on the space of all such surfaces. We further proceed to precisely calculate the classification difficulty of this equivalence relation in terms of the modern theory of Borel equivalence relations.

On the other hand we show that the analog of Becker, Rubel, and Henson's question has a negative solution in (complex) dimension $n \geq 2$.

1. Introduction

In this paper we consider the problem of classifying various classes of complex manifolds. The investigation is completely abstract, since we are not so much con-

example III

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Theorem (Hjorth–Kechris, 2000)

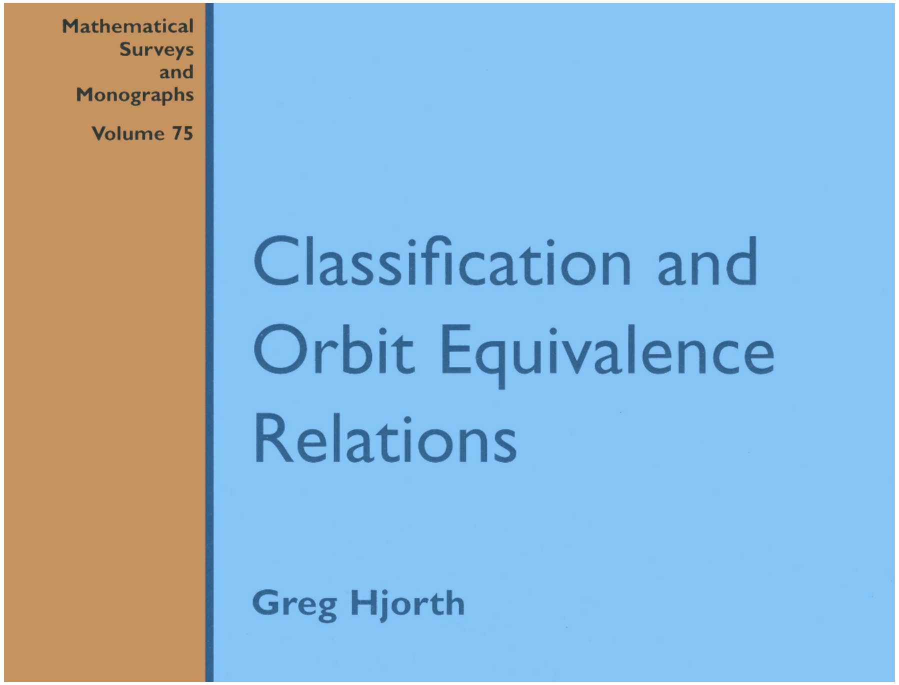
1. *The classification of Riemann surfaces up to biholomorphism is Borel equivalent to the degree* E_∞.
2. *For any* $n \geq 2$, *the biholomorphism relation on complex* n*-manifolds does not admit classification by countable structures.*

For these results to stand alone in the way that they did seemed to us very strange; to quote our paper, given

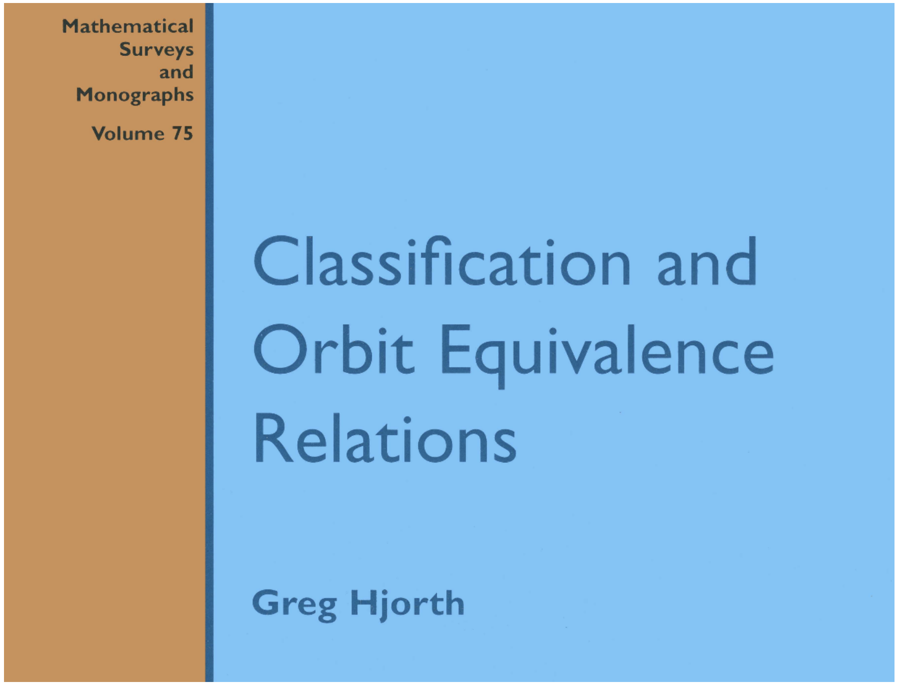
> *not only invariant descriptive set theory's programmatic interest in classification problems throughout mathematics (particularly in manifold-adjacent fields like dynamics [FRW11, FW22] and geometric group theory [Tho08, PS21, CC24]), but Hjorth's framing, in one of the field's foundational texts, of the classification of compact surfaces as among the most exemplary of solutions to such problems, and we have the makings of a positive mystery.*

example IV

example IV


Mathematical
Surveys
and
Monographs
Volume 75
Classification and
Orbit Equivalence
Relations
Greg Hjorth

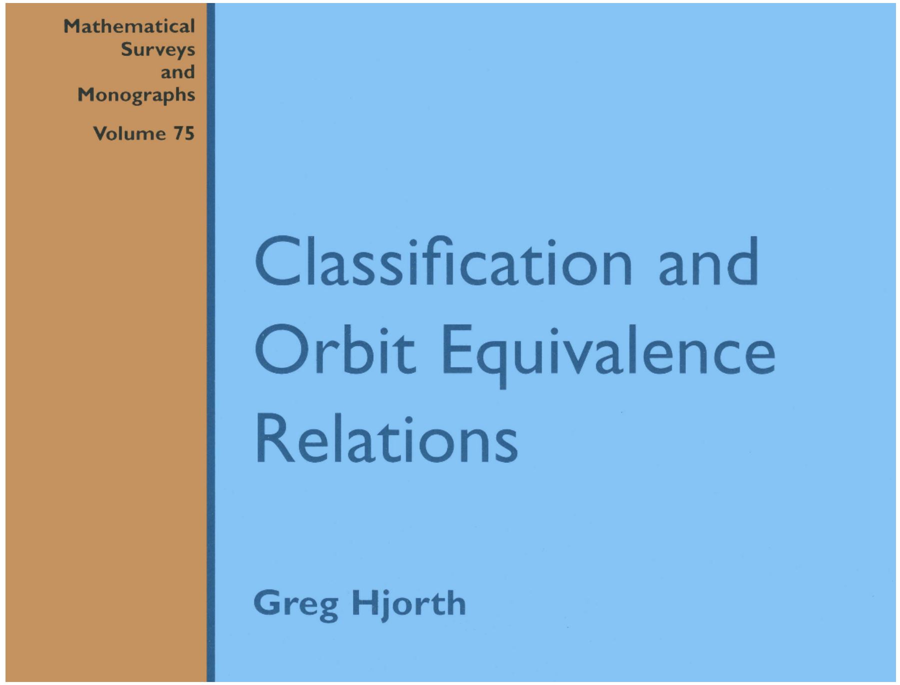
example IV



Preface

What does it mean to *classify* the equivalence classes of some equivalence relation?

example IV



Preface

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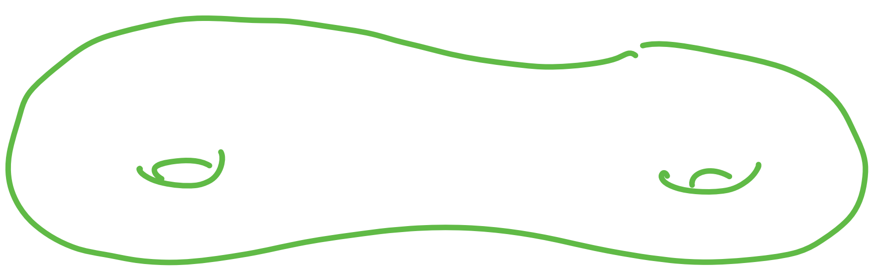
EXAMPLE 0.1. **Low dimensional topology** For compact orientable surfaces a complete invariant can be obtained by simply counting the number of handles, and moreover, this invariant can be produced in a *recursive* or *computable* fashion from a finite triangularization of the manifold; for non-orientable surfaces the

example IV in pictures

surfaces	genus
	0
	1
	2
⋮	⋮

"naive invariant descriptive set theory"

$$\left\{ \begin{array}{c} \text{closed surfaces} \\ \text{up to homeo} \end{array} \right\} =_B \; =_{\mathbb{N}}$$

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Theorem (Markov, 1958)

There exists an effective map M from the class of finite presentations P of groups $G(P)$ to that of compact 4-manifolds such that:

1. $\pi_1(M(P)) = G(P)$ *for all* P, *and*
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This M reduces the recursively unsolvable *group isomorphism problem* to *the problem of classifying compact 4-manifolds up to homeomorphism*, and carries the folklore corollary that the latter is "impossible", but this inference should, obviously, be handled with care. . .

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For every $n > 0$, both the classification

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In particular, from an invariant descriptive set-theoretic perspective, the classification of compact 4-manifolds up to homeomorphism amounts to the *simplest* of all infinite classification problems. This is, of course, compatible with a view, deriving from Markov's theorem, of the classification of compact 4-manifolds as *hard*, but only alongside a view of classification problems generally accredited as solved (e.g., examples I and II) as being even harder.

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And any better than a naive comparison of their associated classification problems' complexities calls for *parameter spaces* for each of these sorts of manifolds.

Readymade and well-known ones do, for many classes, exist. More generally, though, one's faced with repeated assignments of exercises like the following:

Exercise

Fix $n > 1$ *and parametrize the class of topological* n*-manifolds in a standard Borel way.*

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> n-dimensional complex manifold M there is at least one $p \in \mathcal{M}^n$ with $M \cong M_p$. We call $\mathcal{M}^n$ the *parameter space of n-dimensional complex manifolds*. The construction of $\mathcal{M}^n$ and the verification that it has a number of reasonable properties that we will need in various parts of this paper is technically cumbersome, although mathematically rather shallow. We will thus postpone the precise definition and verification

business of parametrizing its objects of study.

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Theorem B

Any Borel pseudogroup $\mathcal{G}$ *on a locally compact Polish space* X *induces a standard Borel parameter space* $\mathfrak{M}(\mathcal{G}, X)$ *of all* $(\mathcal{G}, X)$*-manifolds, on which the relation of* $(\mathcal{G}, X)$*-equivalence is an analytic equivalence relation.*

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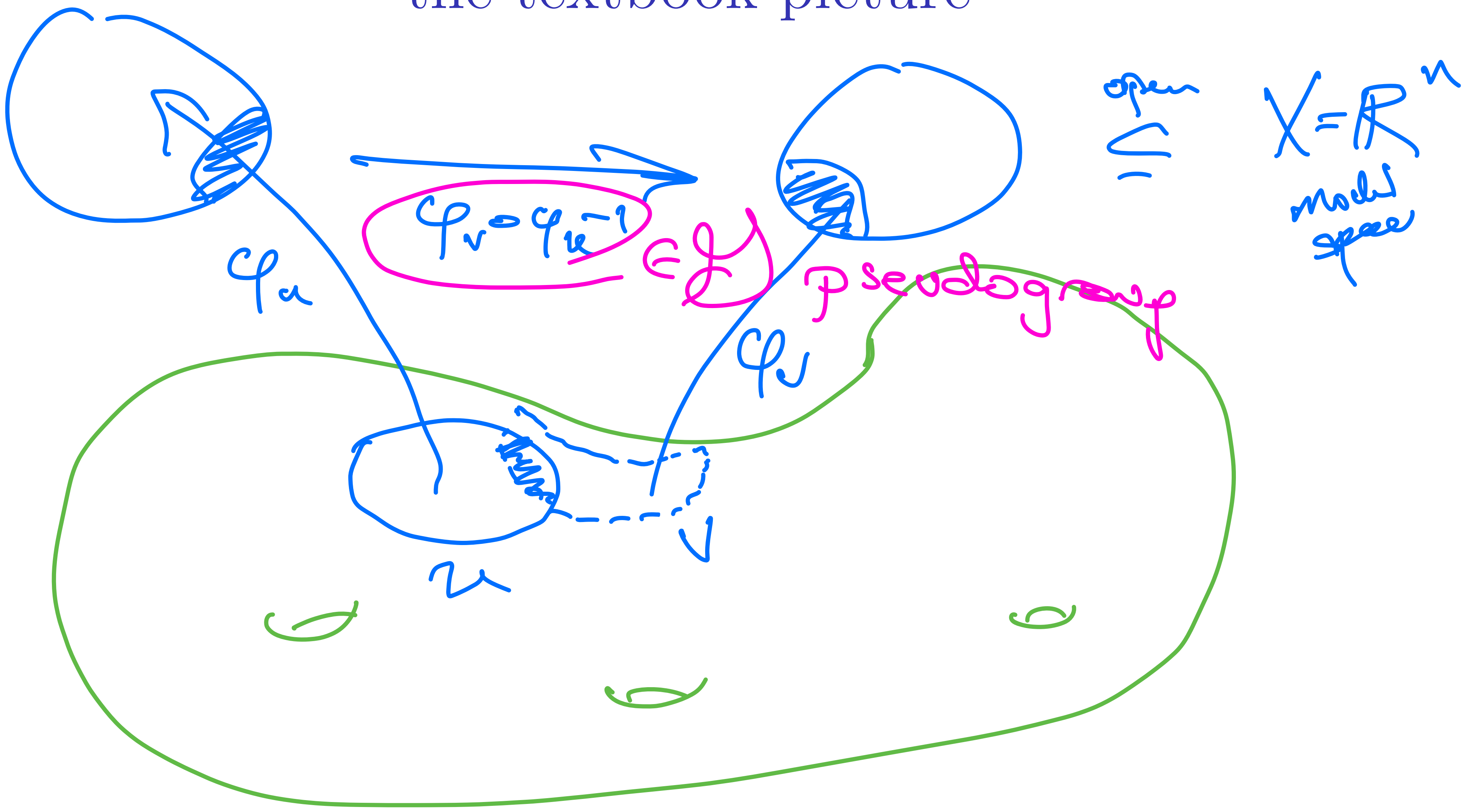
Prominent examples of Borel pseudogroups include Top and C^∞ on $\mathbb{R}^n$, and Isom on $\mathbb{H}^n$, determining standard Borel parameter spaces of all topological, smooth, and hyperbolic n-manifolds, respectively. Let's define our terms.

the textbook picture



pseudogroups

pseudogroups

Definition

A *pseudogroup* $\mathcal{G}$ on a topological space X is a set of homeomorphisms between open subsets of X satisfying the following conditions:

1. the domains of the elements of $\mathcal{G}$ cover X,
2. the restriction of any element of $\mathcal{G}$ to an open subset of X is also in $\mathcal{G}$,
3. the composition of any two elements of $\mathcal{G}$ is in $\mathcal{G}$,
4. the inverse of any element of $\mathcal{G}$ is in $\mathcal{G}$, and
5. if $\varphi : U \to V$ is a homeomorphism between open subsets U and V of X, and U is covered by a collection of open sets U_α with the property that the restriction $\varphi|_{U_\alpha}$ is in $\mathcal{G}$ for every α, then φ is in $\mathcal{G}$.

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Any collection of homeomorphisms between open subsets of X falls in some unique minimal pseudogroup, which it may thus be said to generate.

In particular, any group G of homeomorphisms of X may then be said to generate a pseudogroup, which we denote by a sans-serif G; these form a prominent class of examples.

Definition

Let X denote $\mathbb{R}^n$, $\mathbb{C}^n$, the n-dimensional unit sphere $\mathbb{S}^n$, or hyperbolic space $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$, with the latter two endowed with their standard spherical and hyperbolic metrics, respectively.

1. Top is the *topological pseudogroup* on X, the collection of all homeomorphisms between open subsets of X. This is the maximal pseudogroup on X referred to above.
2. C^k $(1 \leq k \leq \infty)$ is the pseudogroup of all C^k-diffeomorphisms between pairs of open subsets of X.
3. In the case of $X = \mathbb{C}^n$, Hol is the pseudogroup of all biholomorphic maps between open subsets of $\mathbb{C}^n$.
4. Isom is the pseudogroup generated by the group of isometries of X, and Isom^+ is that generated by the orientation-preserving isometries of X.

As in the above examples, we will often omit mention of the *model space* X when it is clear from context.

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1. A $(\mathcal{G}, X)$-*atlas* on M is a collection of pairs (U, φ), called *charts*, where U is an open subset of M and $\varphi : U \to X$ is a homeomorphism onto its image, such that the sets U cover M, and whenever (U, φ) and (V, ψ) are charts with $U \cap V \neq \varnothing$, the *transition map*

$$\psi \circ \varphi^{-1}|_{\varphi[U \cap V]} : \phi[U \cap V] \to \psi[U \cap V]$$

is in $\mathcal{G}$.

The standard definition of a topological n-manifold may now be phrased as: *a second countable Hausdorff space possessing an atlas of charts whose transition maps all lie in the topological pseudogroup on* $\mathbb{R}^n$. To define other sorts of manifolds, we simply vary the pseudogroup appearing therein.

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is in $\mathcal{G}$.

2. M is a *locally* $(\mathcal{G}, X)$-*space* if it is endowed with a $(\mathcal{G}, X)$-atlas; if M is, in addition, Hausdorff and second countable, it is a $(\mathcal{G}, X)$-*manifold.*

Example

The *topological*, *smooth*, and *complex* n-manifolds are precisely the $(\mathsf{Top}, \mathbb{R}^n)$, $(\mathsf{C}^\infty, \mathbb{R}^n)$, and $(\mathsf{Hol}, \mathbb{C}^n)$-manifolds, respectively.

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Manifolds with boundary fit into this framework by letting $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$; the topological and smooth n-manifolds with boundary are then precisely the $(\mathsf{Top}, \mathbb{R}^n_+)$-manifolds and $(\mathsf{C}^\infty, \mathbb{R}^n_+)$-manifolds, respectively.

Definition

Let $\mathcal{G}$ be a pseudogroup on a locally compact Polish space X. The *parameter space* $\mathfrak{P}(\mathcal{G}, X)$ *of second countable locally* $(\mathcal{G}, X)$*-spaces* consists of all pairs $(\mathcal{U}, c)$ where

- $\mathcal{U} = \langle U_{i,j} \mid (i,j) \in \mathbb{N}^2 \rangle$, a family of open subsets of X, and
- $c = \langle \varphi_{i,j} : U_{i,j} \to U_{j,i} \mid (i,j) \in \mathbb{N}^2 \rangle \subseteq \mathcal{G}$

satisfy

1. $U_{i,j} \subseteq U_i := U_{i,i}$ for all $(i,j) \in \mathbb{N}^2$,

together with conditions ensuring that the associated $\sim$ is an equivalence relation:

2. $\varphi_{i,i} = \mathrm{id}_{U_i} : U_i \to U_i$ for all $i \in \mathbb{N}$,
3. $\varphi_{i,j} = \varphi_{j,i}^{-1}$ for all $(i,j) \in \mathbb{N}^2$, and
4. $\varphi_{i,j}^{-1}[U_{j,i} \cap U_{j,k}] \subseteq U_{i,k}$ and $\varphi_{j,k} \circ \varphi_{i,j}\big|_{\varphi_{i,j}^{-1}[U_{j,i} \cap U_{j,k}]} = \varphi_{i,k}\big|_{\varphi_{i,j}^{-1}[U_{j,i} \cap U_{j,k}]}$ for all $i, j, k \in \mathbb{N}$.

Within this framework, it will be useful to term any U_i a *chart*, any $U_{i,j}$ an *overlap*, and any $\varphi_{i,j}$ a *transition function.*

topological and Borel structure

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The *Fell* (or *Chabauty*, or *geometric*) *topology* on the collection $\mathcal{F}(X)$ of closed sets of a locally compact Polish space X is generated by open sets of the form $\{F \in \mathcal{F}(X) : F \cap K = \varnothing\}$ and $\{F \in \mathcal{F}(X) : F \cap U \neq \varnothing\}$, where K and U range over the compact and open subsets of X, respectively.

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For our purposes, the key point is that the basic operations on $\mathcal{O}(X)$ ($(U, V) \mapsto U \cap V$, etc.) are all Borel-measurable.

Borel pseudogroups

Definition

A pseudogroup $\mathcal{G}$ on X is *Borel* if it is endowed with a standard Borel structure such that each of the following maps are Borel on the relevant spaces:

1. domain projection $\mathcal{G} \to \mathcal{O}(X) : \varphi \mapsto \mathrm{dom}(\varphi)$,
2. range projection $\mathcal{G} \to \mathcal{O}(X) : \varphi \mapsto \mathrm{ran}(\varphi)$,
3. domain inclusion $\mathcal{O}(X) \to \mathcal{G} : U \mapsto \mathrm{id}_U$, the identity map on U,
4. direct image $\mathcal{O}(X) \times \mathcal{G} \to \mathcal{O}(X) : (U, \varphi) \mapsto \varphi[U]$,
5. inverse image $\mathcal{O}(X) \times \mathcal{G} \to \mathcal{O}(X) : (U, \varphi) \mapsto \varphi^{-1}[U]$,
6. restriction $\mathcal{O}(X) \times \mathcal{G} \to \mathcal{G} : (U, \varphi) \mapsto \varphi|_U$,
7. composition $\mathcal{G} \times \mathcal{G} \to \mathcal{G} : (\varphi, \psi) \mapsto \psi \circ \varphi$, and
8. inversion $\mathcal{G} \to \mathcal{G} : \varphi \mapsto \varphi^{-1}$.

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Theorem

If $\mathcal{G}$ *is a pseudogroup on a locally compact Polish space* X *then* $\mathfrak{P}(\mathcal{G}, X)$ *inherits from* $\mathcal{O}(X)^{\mathbb{N}\times\mathbb{N}} \times \mathcal{G}^{\mathbb{N}\times\mathbb{N}}$ *a standard Borel structure.*

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The parameter space $\mathfrak{M}(\mathcal{G}, X)$ of $(\mathcal{G}, X)$-manifolds is the set of pairs $(\mathcal{U}, c)$ in $\mathfrak{P}(\mathcal{G}, X)$ such that $M_{(\mathcal{U},c)}$ is a $(\mathcal{G}, X)$-manifold, or, equivalently, is Hausdorff; this defines a Borel subset of $\mathfrak{P}(\mathcal{G}, X)$.

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The pseudogroups Top *and* C^∞ *on $\mathbb{R}^n$, and* Isom *on $\mathbb{H}^n$ are all Borel.*

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This also tells us that the locus of more substantial complexity must be *noncompact* manifolds, or, more individually, their noncompact parts, their *ends*. This brings us back to examples I and II.

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As noted, for anyone familiar with the Kerékjártó–Richards theorem, this computation's less than surprising. Already in 2011, Clinton Conley had noted (on MathOverflow) the connection to Carmelo and Gao's result [CG01] that the homeomorphism relation $\cong$ on the space $\mathcal{K}(\{0,1\}^{\mathbb{N}})$ of closed subsets of the Cantor space $\{0,1\}^{\mathbb{N}}$ is complete for countable structures.

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Corollary 1.3. *The equivalence relation of homeomorphism on the space of surfaces with a pants decomposition is bireducible with C_∞.*

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There is a Borel measurable function which takes any embedding γ of S^1 into $\mathbb{R}^2$ to a self-homeomorphism of $\mathbb{R}^2$ which extends γ.

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And it was at this point clearer why no one before had recorded the Theorem C which we'd for a moment suspected of being essentially routine: it hadn't at all turned out that way for us.

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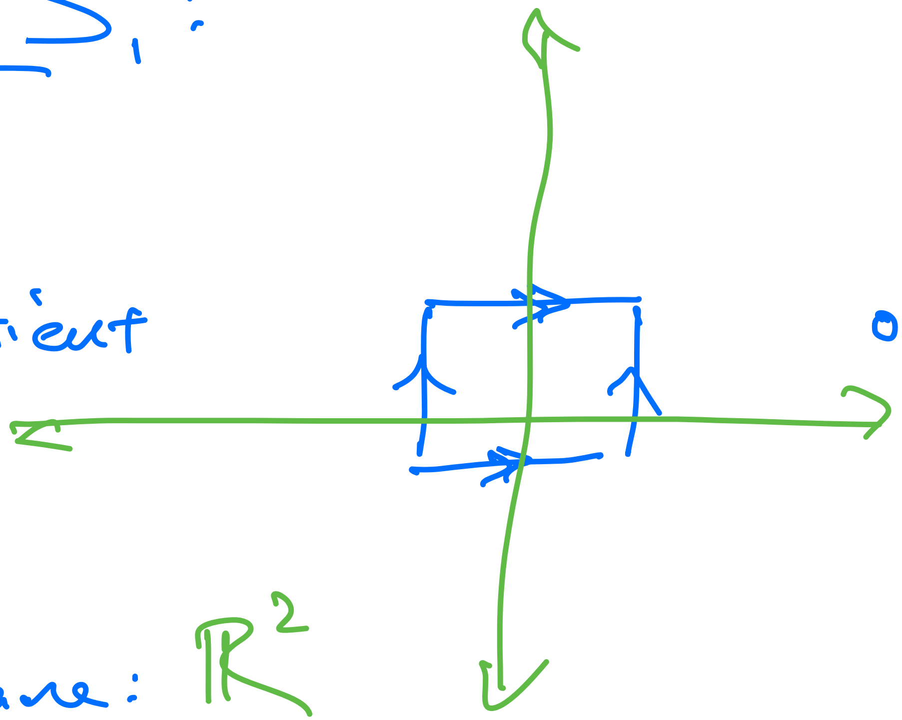
Nevertheless, almost any of the questions now arising to your mind are, to the best of my knowledge, open.

There's much to be done.

more geometric questions

the torus S_1:

a quotient of a square, or, better,



of a plane: $\mathbb{R}^2$

by the action of a discrete group $\mathbb{Z}^2 \leq \mathrm{Isom}(\mathbb{R}^2)$

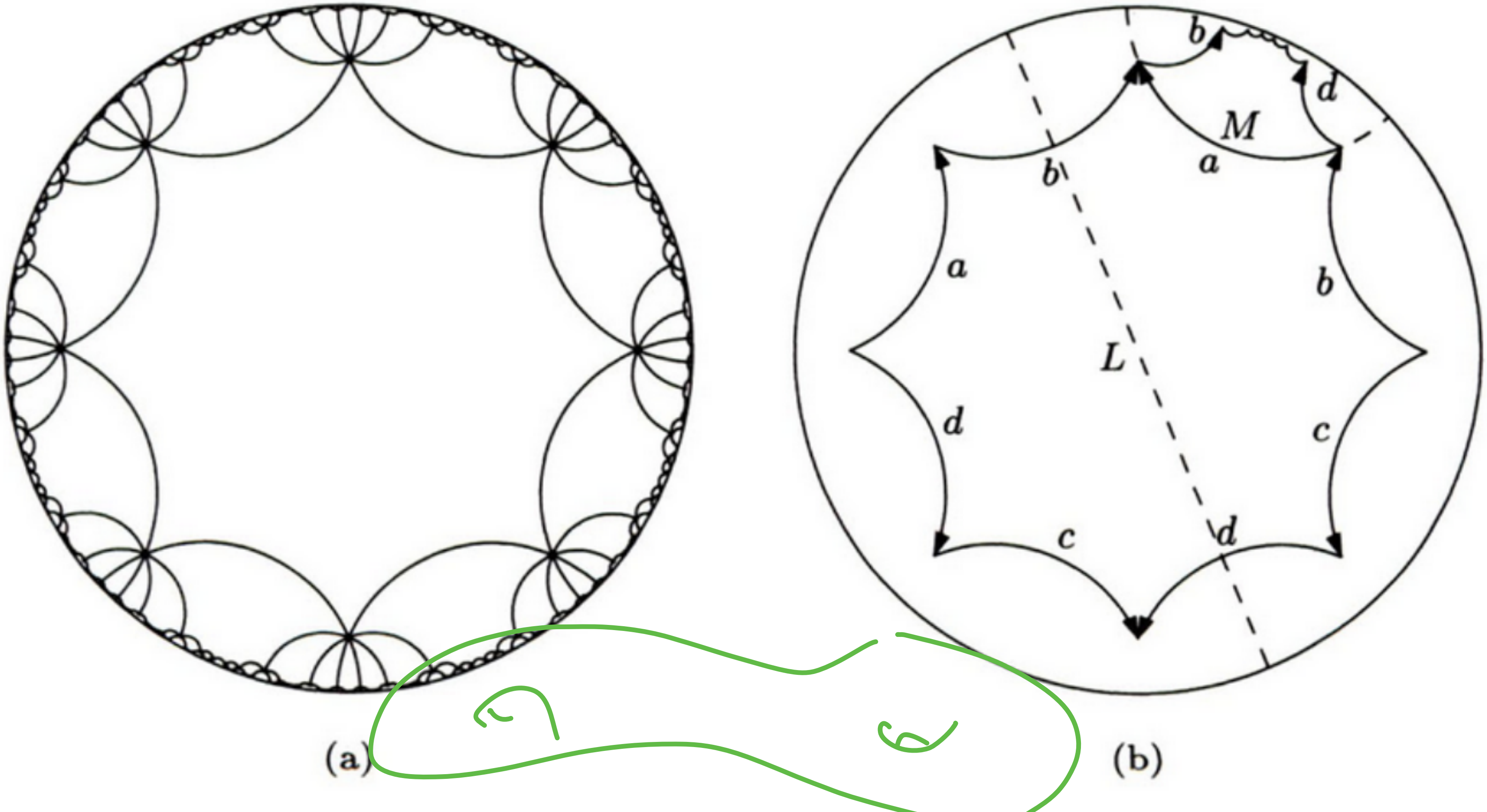
with the square figuring as ~~the~~ a

fundamental domain for the action.

$\underline{S_2}$: can glue together two $S_{1,1}$ - but S_2 should also have an octagonal fund domain, with interior angles all $\pi/4$.

- Obviously can't tile $\mathbb{R}^2$ with that (or even find just one such) - but we can, in $\mathbb{H}^2$.

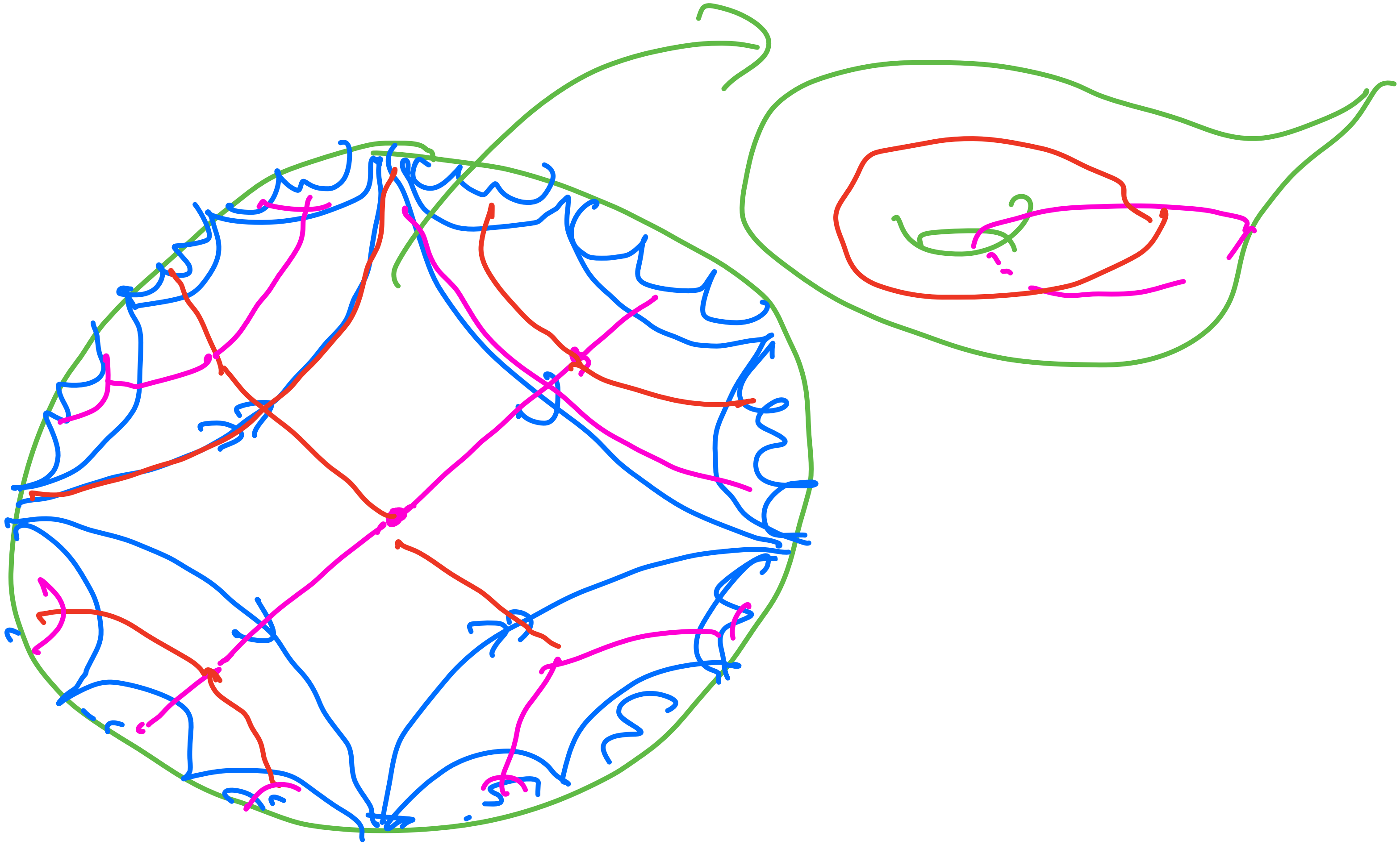
Recall the Poincaré disc model of $\mathbb{H}^2$



(a) (b)

Figure 1.13. A tiling of the hyperbolic plane by regular octagons. (a) A tiling of the hyperbolic plane by identical regular octagons, seen in the Poincaré disk projection. (b) To get the small octagon from the big one, reflect in L, then in M.

$S_{1,1}$



$$\pi_1(S_{1,1}) = \langle a, c \rangle$$

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here I want to draw out a few points:

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First: Complete hyperbolic 2-manifolds M are quotients of $\mathbb{H}^2$ by discrete images of $\pi_1(M)$ in $\mathrm{Isom}(\mathbb{H}^2)$

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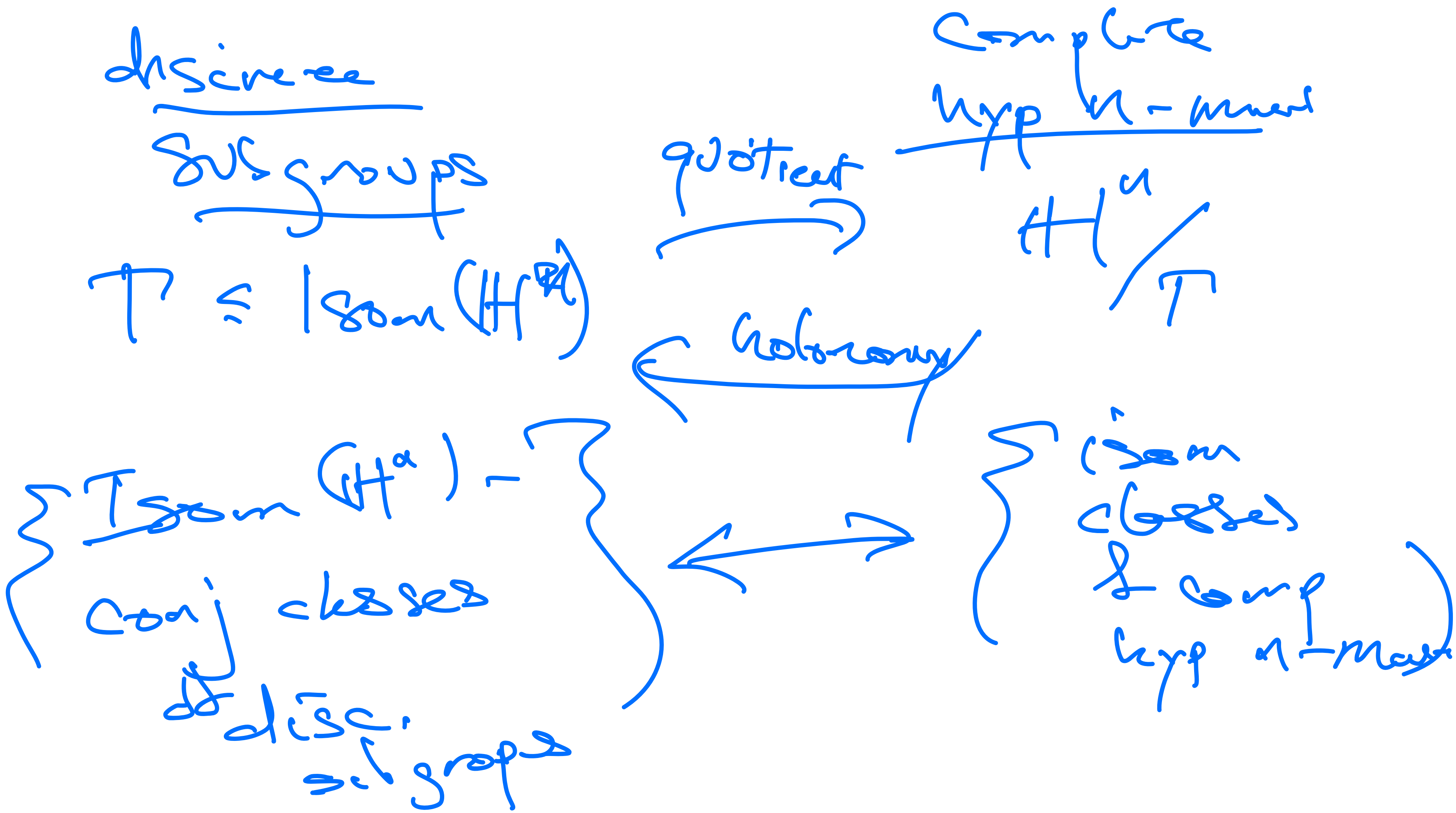

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The conjugacy relation on discrete subgroups of any noncompact semisimple Lie group G is Borel equivalent to E_∞. In particular, this holds for $G = \mathrm{Isom}(\mathbb{H}^n)$, and thus for the isometry relation on complete hyperbolic n-manifolds as well.

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We write $E(G, \mathcal{D}(G))$ for the induced equivalence relation. Note also for future reference that, for G as above, the space $\mathcal{D}_{\mathrm{fg}}(G)$ of finitely generated discrete subgroups of G is a Borel subset of $\mathcal{D}(G)$.

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- In [ACH01], Andretta, Camerlo, and Hjorth showed that $E(G, \mathcal{D}(G))$ is Borel bireducible with E_∞, for any countable group G containing F_2.

(as background)

Let F_2 denote the free group on two generators.

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Theorem F, above, both draws on these results and rounds out the picture; its more concise phrasing is:

For any noncompact semisimple Lie group G, the relation $E(G, \mathcal{D}(G))$ is Borel bireducible with E_∞.

One upshot of this result is that

Question

Given a noncompact semisimple Lie group G, what is the Borel complexity of $E(G, \mathcal{D}_{\mathrm{fg}}(G))$?

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Fix $n \geq 2$. What is the Borel complexity of the isometry relation on the space of connected complete hyperbolic n-manifolds with finitely generated fundamental group?

That last part's a mouthful. We'll call such manifolds *algebraically finite*; they satisfy the most general of the finiteness conditions we might impose.

finiteness conditions

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Fix $n \geq 2$ and let $M = \mathbb{H}^n/\Gamma$ be the quotient hyperbolic manifold associated to a torsion-free discrete subgroup Γ of $\mathrm{Isom}(\mathbb{H}^n)$.

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These conditions are of (weakly) decreasing stringency: for each n, the classes of n-manifolds associated to these conditions are nondecreasing in size. For dimensions $n = 2$ and 3, we compute the complexity of the isometry relation on each of them.

two hyperbolic dimensions

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Theorem G

The orientation-preserving isometry problem for any of the classes (1) through (5) of complete orientable hyperbolic 2-manifolds listed above is Borel bireducible with $=_{\mathbb{R}}$*. In particular, the classification of*

1. *finitely generated torsion-free Fuchsian groups up to conjugacy, or, equivalently, of*
2. *algebraically finite complete orientable hyperbolic 2-manifolds up to orientation-preserving isometry*

is Borel bireducible with $=_{\mathbb{R}}$*.*

three hyperbolic dimensions

Theorem H

The Borel complexity degrees of the conjugacy relation on the major finiteness classes of torsion-free Kleinian groups are as follows:

- *for lattices, it is* $=_{\mathbb{N}}$*;*
- *for geometrically finite groups, it is* $=_{\mathbb{R}}$*;*
- *for finitely generated groups, it is not concretely classifiable.*

Corresponding results hold for the isometry relation on the corresponding classes of hyperbolic 3-manifolds.

Theorem G

One path to Theorem K is through the Teichmüller and moduli spaces of the relevant surfaces.

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Definition

Let S be a finite-type hyperbolizable surface. We have then the *Teichmüller space*

$$\mathrm{Teich}(S) = \{\text{finite-volume hyperbolic metrics on } S\}/\mathrm{Diff}_0(S),$$

mapping class group

$$\mathrm{MCG}(S) = \mathrm{Diff}^+(S)/\mathrm{Diff}_0(S),$$

and *moduli space*

$$\mathcal{M}(S) = \mathrm{Teich}(S)/\mathrm{MCG}(S)$$

of S.

Theorem G

Informally:

> First, though, we must be more precise about what we mean by two hyperbolic structures being the same. There are in fact two important notions of equivalence, giving rise to two spaces: *moduli space* and *Teichmüller space*. Informally, in Teichmüller space, we pay attention not just to what metric a surface is wearing, but also to how it is worn. In moduli space, all surfaces wearing the same metric are equivalent. The importance of the distinction will be clear to anybody who, after putting a pajama suit on an infant, has found one leg to be twisted.

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Summing up, the moduli space $\mathcal{M}(S)$ parametrizes finite-volume hyperbolic surfaces homeomorphic to S up to isometry. If S is given by n punctures of a closed genus-g surface then $\mathcal{M}(S)$ is a Polish space of dimension $6g + 2n - 6$.

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Summing up, the moduli space $\mathcal{M}(S)$ parametrizes finite-volume hyperbolic surfaces homeomorphic to S up to isometry. If S is given by n punctures of a closed genus-g surface then $\mathcal{M}(S)$ is a Polish space of dimension $6g + 2n - 6$. By letting S range, we then reduce isometry on finite-volume hyperbolic surfaces to equality on a countable sum of Polish spaces, and a trick allows us to weakly reduce the algebraically finite case to this one.

Theorem H

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Ending Lamination Theorem (Min10, BCM12)

A hyperbolic 3-manifold with finitely generated fundamental group is uniquely determined by its topological type and its end invariants.

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Ending Lamination Theorem (Min10, BCM12)

A hyperbolic 3-manifold with finitely generated fundamental group is uniquely determined by its topological type and its end invariants.

As it happens, these invariants naturally array into a Polish space, which is promising. Upon closer inspection, however, the theorem classifies *marked* hyperbolic manifolds, and passage from these to our *unmarked* domain of concern is via a quotient by a group action *not*, in general, so nice as that of $\mathrm{MCG}(S)$ on $\mathrm{Teich}(S)$.

Theorem I

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The relevant terms here are best defined, under the circumstances, via pictures.

limit sets

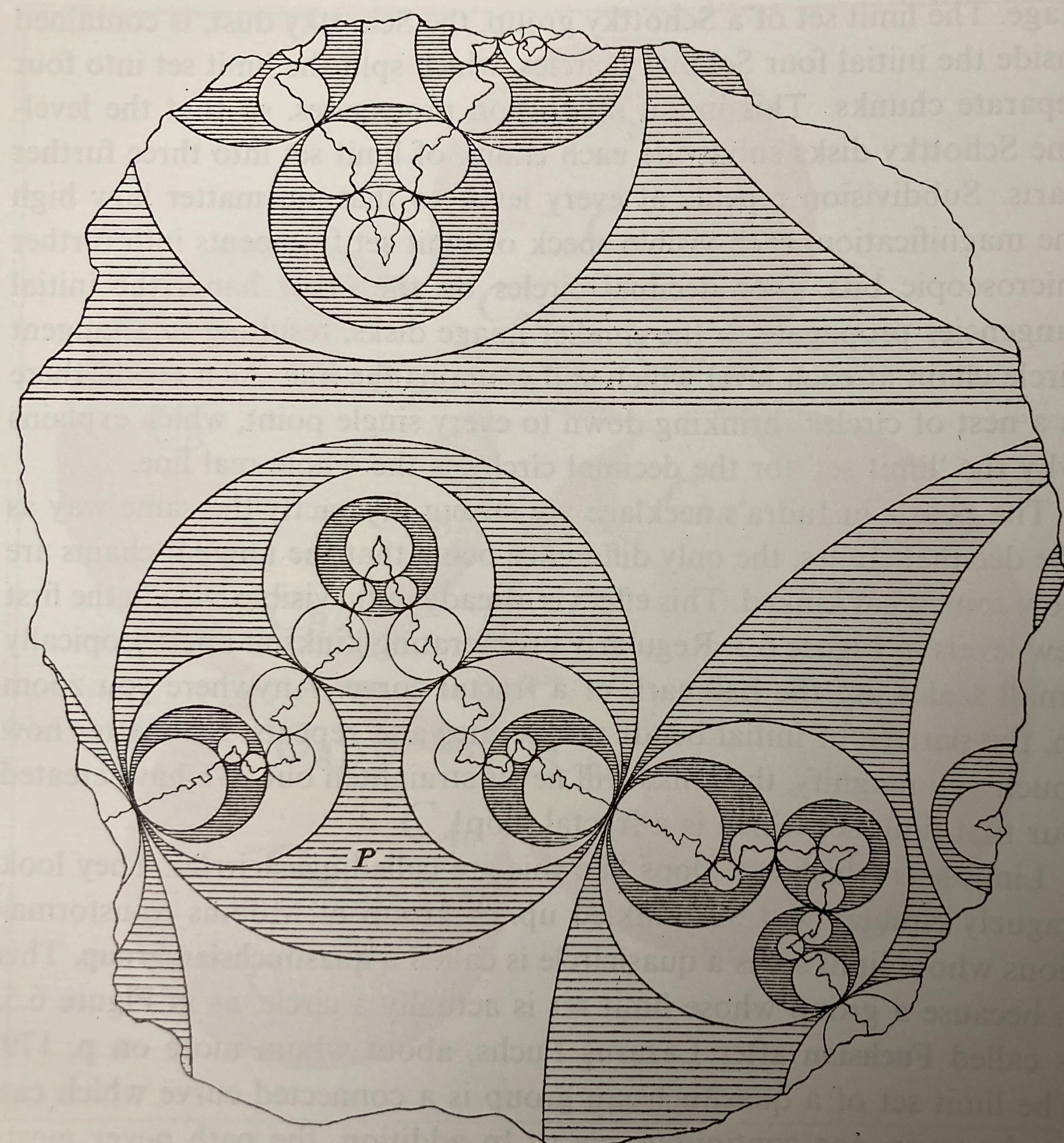


Figure 6.4. A reproduction of Figure 145 in Fricke and Klein, *Vorlesungen über die Theorie der Automorphen Functionen*, Leipzig, 1897, Vol. 1, p. 418, showing the limit set of a Schottky group made by pairing a chain of tangent circles.

limit sets

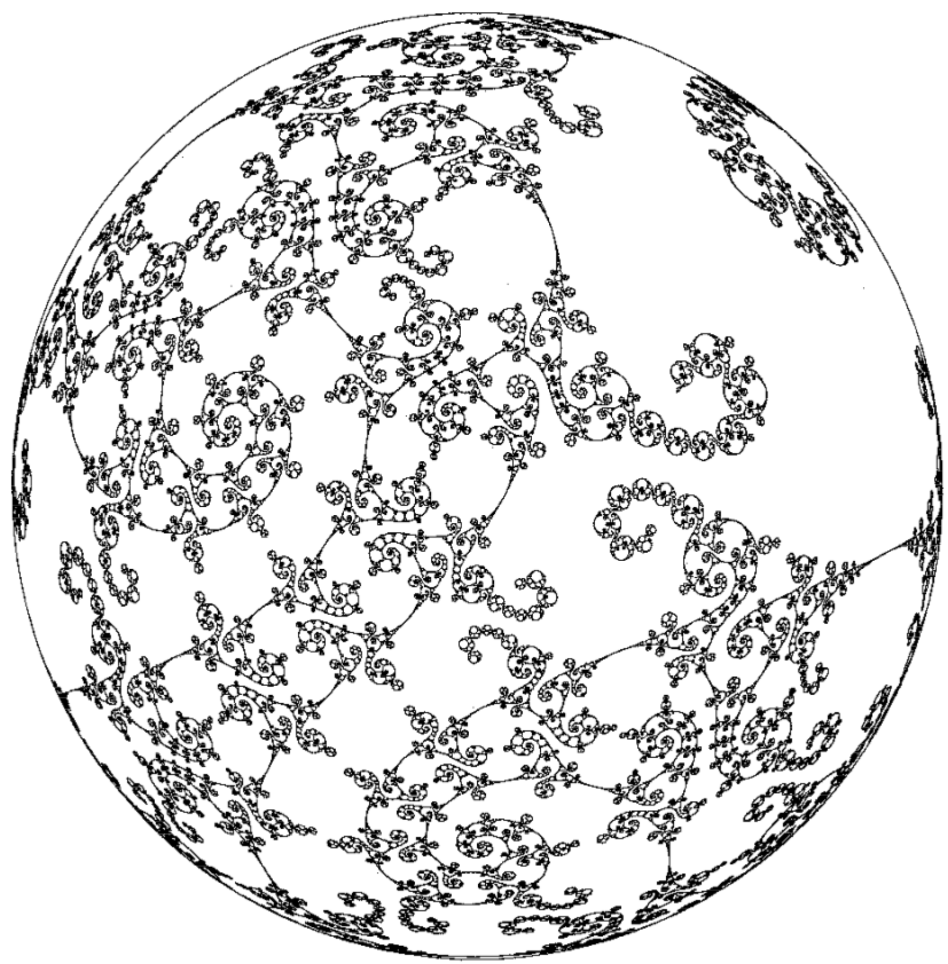


FIGURE 1. This is a portion of the limiting sphere-filling curve for a fiber of the punctured torus bundle over S^1 with gluing map R^4L, where R is a right-handed Dehn twist about a $(1,0)$-curve and L is a left-handed Dehn twist around a $(0,1)$-curve.

limit sets

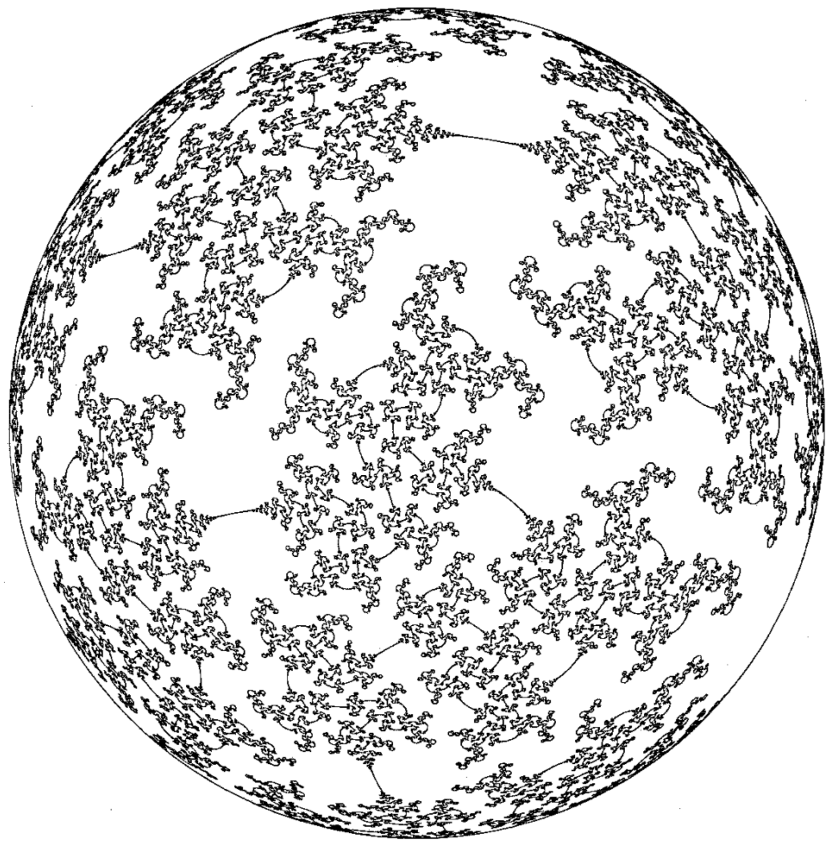


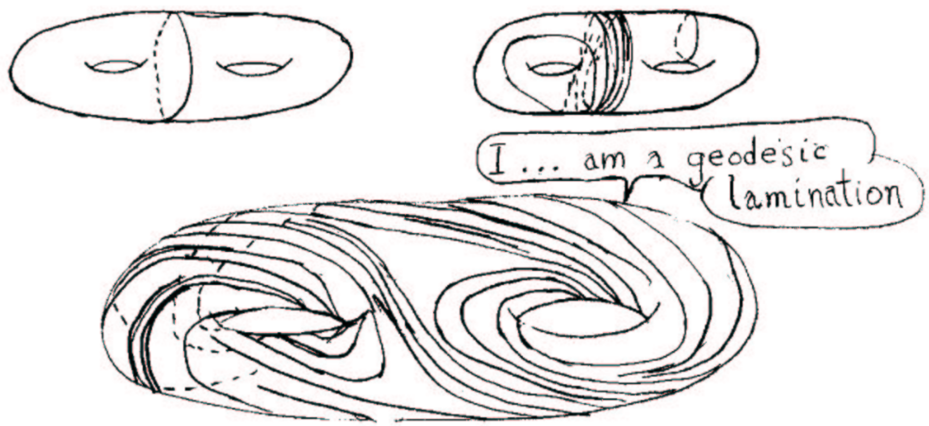
FIGURE 2. An approximation to a sphere filling curve which arises from the fiber of a hyperbolic three-manifold which fibers over the circle. The three-manifold in this case is the complement of the figure eight knot, and the fiber is the punctured torus bounded by the figure eight knot.

Theorem 0.2 (Sphere filling curve; Cannon and Thurston)**.** *Let M^3 be a hyperbolic 3-manifold which fibers over S^1 with fiber F. Then the map $i : \tilde{F} \to \tilde{M}$ extends continuously to a map*

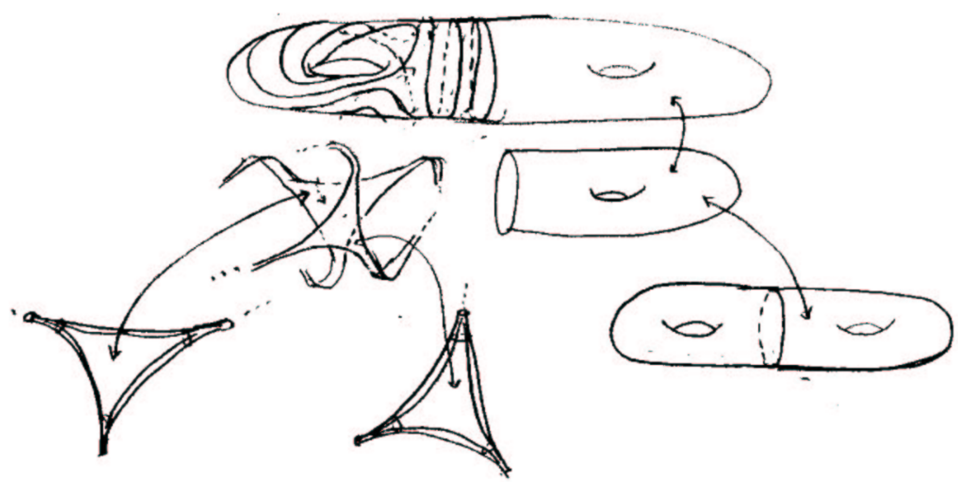
$$\bar{i} : D^2 \to D^3.$$

The boundary of D^2 thus gives a sphere filling curve, or Peano curve, on S^2_∞.

laminations



By consideration of Euler characteristic, the lamination γ cannot have all of ∂M as its support, or in other words it cannot be a foliation. The complement $\partial M - \gamma$ consists of regions bounded by closed geodesics and infinite geodesics. Each of these regions can be doubled along its boundary to give a complete hyperbolic surface, which of course has finite area. There



Proof 1

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Proof 1 of Theorem I uses Przytycki–Sabok's result that orbit equivalence relation induced by the natural action of $\mathrm{MCG}(S)$ on a space homeomorphic to that of the ending laminations of S is of Borel complexity exactly E_0.

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UNICORN PATHS AND HYPERFINITENESS FOR THE MAPPING CLASS GROUP

PIOTR PRZYTYCKI† AND MARCIN SABOK‡

Abstract. Let S be an orientable surface of finite type. Using Pho-On's infinite unicorn paths, we prove the hyperfiniteness of orbit equivalence relations induced by the actions of the mapping class group of S on the Gromov boundaries of the arc graph and the curve graph of S. In the curve graph case, this strengthens the results of Hamenstädt and Kida that this action is universally amenable and that the mapping class group of S is exact.

Proof 2

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Proof 2 of Theorem I leverages a coarse embedding of the shift space into the space of hyperbolic 3-manifolds (due to Abert–Bergeron–Biringer–Gelander–Nikolov–Raimbault–Samet).

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Before beginning the proof in earnest, we give a motivational outline. The idea is to associate to use the Schottky group $\langle \phi_1, \ldots, \phi_n \rangle$ to associate to every element $\gamma \in \{0, \ldots, n\}^{\mathbb{Z}}$ of the shift space a pair consisting of the following elements:

(1) a hyperbolic 3-manifold N_γ homeomorphic to $\Sigma \times \mathbb{R}$
(2) a 'coarse base point' P_γ, i.e. a subset of N_γ with universally bounded diameter.

Shifting a string γ corresponds to shifting the base point of N_γ and convergence of $\gamma_i \in \{0, \ldots, n\}^{\mathbb{Z}}$ corresponds to based Gromov Hausdorff convergence of the associated pairs $(N_{\gamma_i}, P_{\gamma_i})$. A periodic string with period $(e_1, \ldots, e_k)$ corresponds

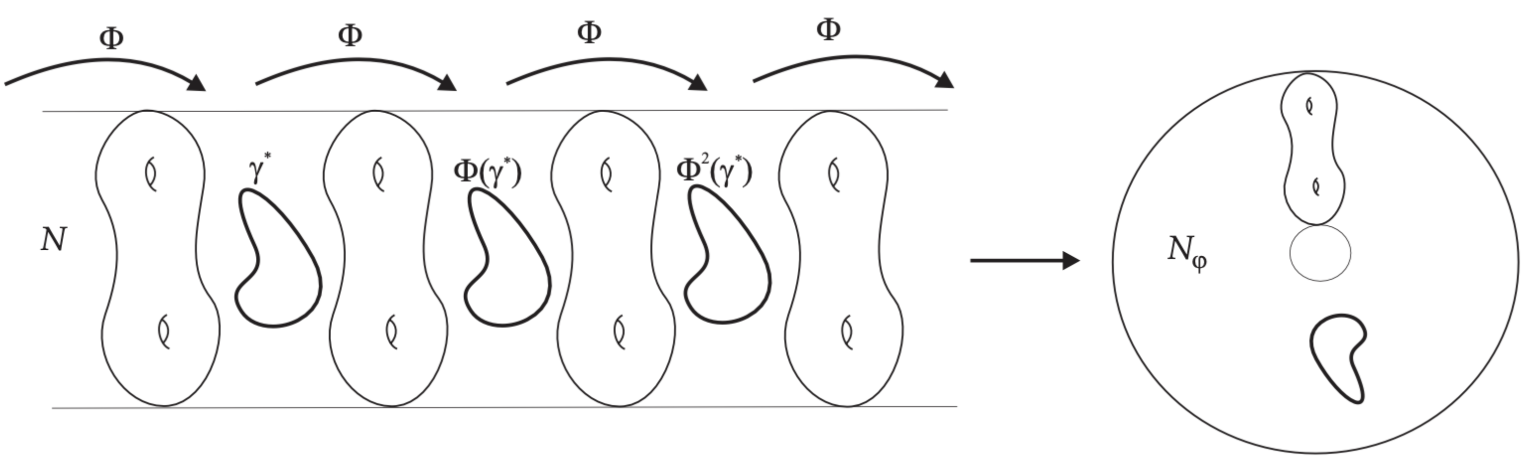

FIGURE 7. N covers the surface bundle N_φ.

in closing

To recap: a descriptive set theoretic question about conjugacy relations took the form of one which we answered, in the $n = 2$ and $n = 3$ cases, using fundamentally geometric tools:

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many thanks

Thanks to Professor Kechris for the invitation,

and to the audience for their attention,

and for any questions which any of you may have.