Rohlin's lemma in the infinite measure-preserving setup

(with Fabien Hoareau)

<u>Setup</u>: (X,λ) std ∞ measured space ($\cong (\mathbb{R}, \mathrm{leb})$)

(X std Borel + λ σ-finite atomless ∞ measure)

We will also consider std pba spaces, denoted as (X,μ)

$\mathrm{Aut}(X,\lambda) := \{T : X \to X$ Borel bij $\forall A \subseteq X$ Borel, $\lambda(A) = \lambda(T^{-1}(A))\} / \sim_\lambda$

$T_1 \sim_\lambda T_2$ iff $\lambda(\{x \in X : T_1(x) \neq T_2(x)\}) = 0$

A Borel bijectⁿ is aperiodic if it has only ∞ orbits.

<u>Ex</u>: • $(X,\lambda) = (\mathbb{R}, \mathrm{leb})$

$T(x) = x+1$ is aperiodic

It admits a <u>fundamental domain</u> $D = [0,1[$

$\hookrightarrow D \subseteq X$ st $X = \bigsqcup_{n\in\mathbb{Z}} T^n(D)$

In such a situation, we say T is <u>dissipative</u>.

Say that T is <u>conservative</u>: $\forall A \subseteq X$ Borel, $\forall^* x \in A$,

$\exists n \geq 1$ s.t. $T^n(x) \in A$

ex: T periodic (all orbits are finite) $\Rightarrow$ T conservative

<u>Thm</u> (Hopf): Let $T \in \mathrm{Aut}(X,\lambda)$, $\exists!$ T-invariant partitⁿ

$$X = C_T \sqcup D_T$$

st $T_{\restriction C_T}$ is conservative, $T_{\restriction D_T}$ is dissipative.

<u>Ex of conservative bijectⁿ</u>:

• $(X,\lambda) = (\{0,1\}^{\mathbb{N}}, \lambda)$

odometer T_0 : $T_0((x_i)_{i\in\mathbb{N}}) = (x_i) + (1,0,0,0,\dots 0)$

$\hookrightarrow$ additⁿ w/ right carry

$$\begin{array}{l} T_0(1,1,1,0,1,0,\dots) \\ \quad +(1,0,0,0\dots) \\ \hline = (0,0,0,1,1,0,\dots) \end{array}$$

Given $s \in \{0,1\}^{<\mathbb{N}}$,

$$M_s := \{ (x_i) \mid x_1, x_3, \dots x_{2|s|-1} = s \ \ \forall i \geq |s|,\ x_{2i+1} = 0 \} \underset{\text{forgetting odd digits}}{\cong} \{0,1\}^{\mathbb{N}}$$

λ_s = Bernoulli $\frac{1}{2}$-$\frac{1}{2}$ measure on $M_s = \{0,1\}^{\mathbb{N}}$

$$\lambda = \sum_{s \in \{0,1\}^{<\mathbb{N}}} \lambda_s$$

T_0 is ergodic: $\forall A \subseteq X$ Borel st $T(A) = A$, either $\lambda(A) = 0$ or $\lambda(X \setminus A) = 0$

hence conservative

- $(X, \lambda) = \left(\{-1,1\}^{\mathbb{Z}} \times \mathbb{Z},\ \left(\frac{1}{2}(\delta_{-1} + \delta_1)\right)^{\otimes \mathbb{Z}} \otimes \text{counting} \right)$

$$T((x_i), k) = ((x_{i-1}), k + x_0)$$

Thm: T is ergodic, in particular conservative.

Def: Say $T_1, T_2 \in \mathrm{Aut}(X, \lambda)$ are conjugate if $\exists S \in \mathrm{Aut}(X, \lambda)$ s.t.

$$T_1 = S T_2 S^{-1}$$

Ex: $X = \mathbb{R}$: $T_1(x) = x+1$, $T_2(x) = x-1$, $S : x \mapsto -x$

Thm (Gerber–Kunde '25)
Conjugacy is a non Borel eq. rel on the set of ergodic ∞ MPB.

Thm (Hoareau – LM) Any two aperiodic conservative $T_1, T_2 \in \mathrm{Aut}(X, \lambda)$ are λ-approx. conjugate: $\forall \varepsilon > 0\ \exists S \in \mathrm{Aut}(X, \lambda)$ st $\lambda(\{ T_1(x) \neq S T_2 S^{-1}(x) \}) < \varepsilon$

Rmk:
- Dissipative cannot be λ-approx. conj to conservative
- For dissipative bijections, λ-approx conj is the same as conjugacy

$\rightsquigarrow$ We can classify all ∞ MPB up to λ-approx conjugacy

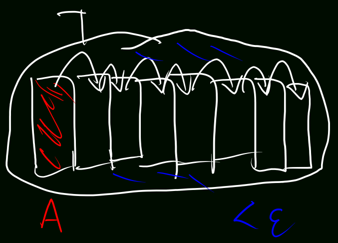
- Take $\mu \sim \lambda$ (proba), then it follows from Rokhlin's lemma that any two aperiodic $T_1, T_2 \in \mathrm{Aut}(X, \lambda)$ are μ-approx conj: $\forall \varepsilon > 0$, $\exists S \in \mathrm{Aut}(X, \lambda)$ st $\mu(\{x \mid T_1(x) \neq S T_2 S^{-1}(x)\}) < \varepsilon$

Rohlin's lemma:

↳ Let $T \in Aut(X,\mu)$ aperiodic, $\varepsilon > 0$, $N \in \mathbb{N}$, $\kappa > 0$ st $1-\varepsilon < N\kappa < 1$

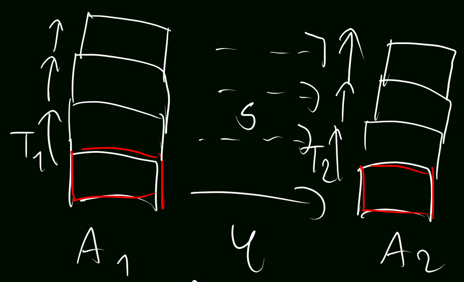
prob case then $\exists A \subseteq X$, $\mu(A) = \kappa$ and $A, T(A), \dots, T^{N-1}(A)$ are disjt

$\left(\Rightarrow \mu(X \setminus A \sqcup T(A) \sqcup \dots \sqcup T^{N-1}(A)) < \varepsilon \right)$



$\rightsquigarrow T_1, T_2 \in Aut(X,\mu)$ aperiodic $\varepsilon > 0$, $\kappa = 1 - \frac{\varepsilon}{2}$

N st $\frac{1}{N} < \varepsilon/2$



fix $\varphi : A_1 \to A_2$ mpB ... extend to S naturally

$\square$

↳ $T \in Aut(X,\lambda)$ aperiodic, $\varepsilon > 0$, $N \in \mathbb{N}$, $\kappa > 0$, $\underline{B \subseteq X}$ of finite measure

∞ MP

then $\exists A \subseteq X$, $\lambda(A) = \kappa$.

$\lambda(B \setminus A \sqcup T(A) \dots \sqcup T^{N-1}(A)) < \varepsilon$

About the proof:

Lemma: $\left[\forall T \text{ aperiodic conservative } \forall \varepsilon > 0, \exists B \subseteq X \right.$
$\lambda(B) < \varepsilon$ & B intersects all T orbits.

$\rightsquigarrow$ Partition of B according to return times ($x \overset{B}{\mapsto} \tau_{T,B}(x) = \min\{n \geq 1$
$T^n(x) \in B\}$)

$B = \bigsqcup_{n \geq 1} \tau_{B,T}^{-1}(\{n\})$

$\rightarrow \sum_{n \geq 1} \lambda(\tau_{B,T}^{-1}(n)) = \lambda(B)$

$\| X = \bigsqcup_{n \geq 1} \bigsqcup_{i=0}^{n-1} T^i(\tau_{B,T}^{-1}(\{n\}))$

$\rightarrow \sum_{n \geq 1} n \lambda(\tau_{B,T}^{-1}(n)) = +\infty = \lambda(X)$

Key lemma

Let (λ_n) st $\sum \lambda_n < +\infty$
$\forall n \geq 1, \quad \lambda_n > \lambda(\tau_{B,T}^{-1}(n))$

$\exists C$ st $\forall n \geq 1,$
$$\tau_{B \sqcup C}^{-1}(n) = \lambda_n$$

$T_1, T_2 \rightsquigarrow B_1, B_2$

$$\forall n \geq \quad \lambda(\tau_{B_1,T_1}^{-1}(n)) = \lambda(\tau_{B_2,T_2}^{-1}(n))$$

$$\varphi : B_1 \to B_2$$
$$\varphi(\tau_{B_1,T_1}^{-1}(n)) = \tau_{B_2,T_2}^{-1}(n)$$

Q° : What about elementary conservative free act° of a f.g. ∞ amenable group Γ?