Topological Groups with Tractable Minimal Dynamics

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G - topological group.

G-flow - A continuous action of G on a compact Hausdorff space X.

Subflow - A non-empty, closed, G-invariant subspace of X.

G-map - A continuous, G-equivariant map from one G-flow to another.

Minimal - Every orbit is dense. Equivalently, no proper subflows.

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The goal of this talk is to map out some dividing lines in the possible complexity of $\mathrm{M}(G)$ and define classes of topological groups based on their minimal dynamics.

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Theorem (Kechris, Pestov, Todorcevic 2005 [15])

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The precise statement of the Ramsey property isn't so important for this talk, but it is a purely combinatorial statement. In particular, we note that it is Δ_1.

For Polish groups, there is a natural next level of generality beyond extremely amenable, namely having $\mathrm{M}(G)$ metrizable.

PCMD - Polish groups with concrete minimal dynamics

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Theorem (KPT 2005 [15], Nguyen Van Thé 2013 [18])

Let $\mathcal{K}$ be a Fraïssé class of L-structures, with $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$, $G = \mathrm{Aut}(\mathbf{K})$. Suppose that there is a Fraïssé class $\mathcal{K}^*$ of L^*-structures, $L^* \supseteq L$, making $(\mathcal{K}^*, \mathcal{K})$ an excellent pair. Then $\mathrm{M}(G) \cong X_{\mathcal{K}^*}$, the space of $\mathcal{K}^*$-expansions of $\mathbf{K}$.

$\mathrm{M}(S_\infty) = \mathrm{LO}(\omega)$, $\mathrm{M}(\mathrm{Homeo}(2^\omega)) \subseteq \mathrm{LO}(\mathrm{Clop}(2^\omega))$.

Theorem (Z. 2016 [22])

If $\mathcal{K}$ is a Fraïssé class with $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$ and $G = \mathrm{Aut}(\mathbf{K})$, then $G \in \mathsf{PCMD}$ iff there is $\mathcal{K}^*$ making $(\mathcal{K}^*, \mathcal{K})$ an excellent pair.

Consequence: If G is as above, then $\mathrm{M}(G)$ has a comeager orbit, namely the orbit of $\mathrm{Flim}(\mathcal{K}^*)$.

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Theorem (Ben Yaacov, Melleray, Tsankov 2017 [8])

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Theorem (Ben Yaacov, Melleray, Tsankov 2017 [8])

If $G \in \mathsf{PCMD}$, then $\mathrm{M}(G)$ has a comeager orbit.

Combining this with earlier work of Melleray, Nguyen Van Thé, and Tsankov, we obtain:

Theorem (MNVTT 2016 [17], BYMT 2017 [8])

$G \in \mathsf{PCMD}$ iff there is a co-precompact, pre-syndetic, extremely amenable closed subgroup $H \leq G$, in which case $\mathrm{M}(G) \cong \widehat{G/H}$.

Moral - For groups in PCMD, there is a "certificate" that $\mathrm{M}(G)$ is nice (either $\mathcal{K}^*$ or H), as well as an explicit set of instructions for constructing it.

Generic Point Property - G Polish, $\mathrm{M}(G)$ has a comeager orbit. Writing GPP for this class, BYMT 2017 gives $\mathsf{PCMD} \subseteq \mathsf{GPP}$.

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Observation - If $(\mathcal{K}^*, \mathcal{K})$ satisfies all of the properties of an excellent pair except pre-compactness, then we have $G = \mathrm{Aut}(\mathrm{Flim}(\mathcal{K})) \in \mathsf{GPP} \setminus \mathsf{PCMD}$.

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Kwiatkowska [16] gives the first example of $(\mathcal{K}^*, \mathcal{K})$ as above, showing $\mathsf{PCMD} \subsetneq \mathsf{GPP}$. Basso-Tsankov [5] develop many more examples. All of these examples are homeomorphism groups of generalized Ważewski dendrites, or closed subgroups thereof.

Question (Gutman-Tsankov-Z. 2021 [13], Basso-Codenotti-Vaccaro 2024+ [4])

Do we have $\mathrm{Homeo}^+(S^2) \in \mathsf{GPP} \setminus \mathsf{PCMD}$*?*

Theorem (Z. 2021 [24])

$G \in \mathsf{GPP}$ iff there is a pre-syndetic, extremely amenable closed subgroup $H \leq G$, in which case $\mathrm{M}(G) = \mathrm{Sa}(G/H)$.

Here, $\mathrm{Sa}(G/H)$ denotes the Samuel compactification of G/H equipped with its right uniformity. Always $\widehat{G/H} \subseteq \mathrm{Sa}(G/H)$.

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Moral - For groups in $\mathsf{GPP} \setminus \mathsf{PCMD}$, we still have a "certificate" that $\mathrm{M}(G)$ is somewhat nice, but a less explicit set of instructions for building $\mathrm{M}(G)$.

What about non-Polish groups? Bartošová [2, 3, 1] considers automorphism groups of uncountable structures and uses the same Ramsey theorems as in the countable case to say things about these groups.

Example - $M(\mathrm{Sym}(\kappa)) = \mathrm{LO}(\kappa)$.

This space is not metrizable, but still “concrete.” Thus our general definition of concrete minimal dynamics should include this example.

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In previous and present joint work with Basso [6, 7], building on work of Jahel-Z. [14] and Bartošová-Z. (appearing in [23]), we define CMD by any of the following properties:

Theorem (Basso-Z. [6, 7])

For a topological group G, the following are equivalent:

1. *For any G-flow Z, $\mathrm{Min}_G(Z) \subseteq \mathrm{Sub}_G(Z)$ is Vietoris closed.*
2. *For every ultracopower of $\mathrm{M}(G)$, the ultracopower map is an isomorphism.*
3. *$\mathrm{M}(G)$ is G-finite, i.e. if $\{A_n : n < \omega\} \subseteq \mathrm{op}(\mathrm{M}(G))$ and $U \in \mathcal{N}_G$, then $\{UA_n : n < \omega\}$ is not pairwise disjoint.*
4. *For any seminorm σ on G, the pseudo-metric ∂_σ on $\mathrm{M}(G)$ is continuous.*

$\mathrm{op}(X)$ = non-empty open subsets of X.

$\mathcal{N}_G$ a base of symmetric open neighborhoods at e_G.

Definitions for previous theorem:

- The ultracoproduct $\Sigma^G_{\mathcal{U}} X_i$ of $\langle X_i : i \in I\rangle$ along $\mathcal{U} \in \beta I$ by definition satisfies the following universal property – if Y is a G-flow, $Y_i \in \mathrm{Sub}_G(Y)$, and $\varphi_i\colon X_i \to Y_i$ are G-maps, then there is a canonical G-map $\varphi\colon \Sigma^G_{\mathcal{U}} X_i \to \lim_{i\to\mathcal{U}} Y_i$.
  When $X_i = X$ and $\varphi_i = \mathrm{id}_X$, call the associated φ the ultracopower map. See [21].
- A seminorm on G is continous, bounded, symmetric $\sigma\colon G \to [0, +\infty)$ satisfying $\sigma(gh) \leq \sigma(g) + \sigma(h)$.
- Given $\sigma \in \mathrm{SN}(G)$, define ∂_σ on $\mathrm{M}(G)$ via

$$\partial_\sigma(x, y) = \sup\{|f(x) - f(y)| : f \in \mathrm{C}_\sigma(\mathrm{M}(G), [0, 1])\},$$

  where for a G-flow X, $f \in \mathrm{C}_\sigma(X)$ iff $\|f - fg\| \leq \sigma(g)$.

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1. *For any G-flow Z, $\mathrm{Min}_G(Z) \subseteq \mathrm{Sub}_G(Z)$ is meets Hausdorff, i.e. if $X_i \in \mathrm{Min}_G(Z)$ and $X_i \xrightarrow{\mathrm{V}} Y \in \mathrm{Sub}_G(Z)$, then Y has a unique minimal subflow.*
2. *For every ultracopower of $\mathrm{M}(G)$, the ultracopower map is highly proximal.*
3. *$\mathrm{M}(G)$ satisfies the Rosendal criterion: for any $A \in \mathrm{op}(\mathrm{M}(G))$ and $U \in \mathcal{N}_G$, there is $B \in \mathrm{op}(A)$ which is U-TT, i.e. for any $B_0 \in \mathrm{op}(B)$, we have $B \subseteq \overline{UB_0}$.*
4. *For any seminorm σ on G, the pseudometric ∂_σ on $\mathrm{M}(G)$ has a dense set of continuity points.*

Definitions for previous theorem:

- If X is compact Hausdorff, the meets topology on $K(X)$ is generated by subbasic sets of the form $\mathrm{Meets}(A, X) := \{K \in K(X) : A \cap K \neq \emptyset\}$ with $A \in \mathrm{op}(X)$.
- Given G-flows X and Y, a G-map $\varphi\colon X \to Y$ is highly proximal if for any $y \in Y$, there are a net (g_i) from G and $x \in X$ with $\varphi^{-1}[g_i y] \xrightarrow{\mathrm{V}} \{x\}$ for some $x \in X$.

Our abstract investigation of the class TMD yields new results for the class GPP.

Theorem (Basso-Z. [7])

For G Polish, we have $G \in \mathsf{GPP}$ iff $\mathrm{M}(G)$ has a point of first countability.

As a consequence, we obtain the corollary that for G locally compact, non-compact Polish, $\mathrm{M}(G)$ has no points of first countability.

Are there "certificates" for membership in CMD and/or TMD? By proving an "abstract KPT correspondence," we obtain:

Theorem (Basso-Z. [7])

The classes CMD *and* TMD *are* Δ_1.

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Hence we get another characterization of these classes - a topological group is in CMD or TMD, resp., iff in some forcing extension, its two-sided completion is a Polish group in PCMD or GPP, resp.

Proof outline – Let V be the ground model, and let $\mathsf{W} \supseteq \mathsf{V}$ be some outer model, for instance a forcing extension. Given a compact space $X \in \mathsf{V}$, there is a natural way to interpret it as a compact space in a forcing extension W, which we denote X^{W}. If X is a (minimal) G-flow, so is X^{W}.

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Certificate of $G \notin \mathsf{TMD}$ – If $X \in \mathsf{V}$ does not satisfy Rosendal criterion, neither does X^{W}.

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Certificate of $G \notin \mathsf{TMD}$ – If $X \in \mathsf{V}$ does not satisfy Rosendal criterion, neither does X^{W}.

Certificate of $G \notin \mathsf{CMD}$ – If $X \in \mathsf{V}$ is not G-finite, neither is X^{W}.

Positive certificates – We define the notion of a G-skeleton, which associates to each seminorm σ on G a metric space X_σ, along with various bonding maps between the X_σ as σ varies.

Example: Let $\mathcal{K}$ be a Fraïssé class with limit $\mathbf{K}$, and let $\mathcal{K}^*/\mathcal{K}$ be a reasonable expansion class. Given $\mathbf{A} \in [\mathbf{K}]^{<\omega}$, one has the seminorm on $\mathrm{Aut}(\mathbf{K})$

$$\sigma_{\mathbf{A}}(g) = \begin{cases} 0 & \text{if } g|_{\mathbf{A}} = \mathrm{id}_{\mathbf{A}} \\ 1 & \text{else.} \end{cases}$$

Then take $X_{\sigma_{\mathbf{A}}} = \mathcal{K}^*(\mathbf{A})$, the set of expansions of $\mathbf{A}$ in $\mathcal{K}^*$.

We prove that $G \in \mathsf{TMD}$ or CMD iff there is a G-skeleton satisfying various properties (minimality, Ramsey, etc.)

Absoluteness results for $\mathrm{M}(G)$:

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Theorem (Basso-Z. [7])

1. *$G \in \mathsf{CMD}$ iff for any forcing extension $\mathsf{W} \supseteq \mathsf{V}$, we have $(\mathrm{M}(G))^{\mathsf{W}} = \mathrm{M}^{\mathsf{W}}(G)$.*
2. *If $G \in \mathsf{TMD}$ and $\mathsf{W} \supseteq \mathsf{V}$ is a forcing extension, then $\mathrm{M}^{\mathsf{W}}(G)$ is an irreducible extension of $(\mathrm{M}(G))^{\mathsf{W}}$*

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The converse of (2) is open. An affirmative answer to the next question would imply it.

Question

If G Polish and $G \notin \mathsf{GPP}$, does $\mathrm{M}(G)$ have uncountable π-weight? Continuum π-weight?

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