Measurable domatic partitions

(based on undergrad thesis research with Clinton Conley)

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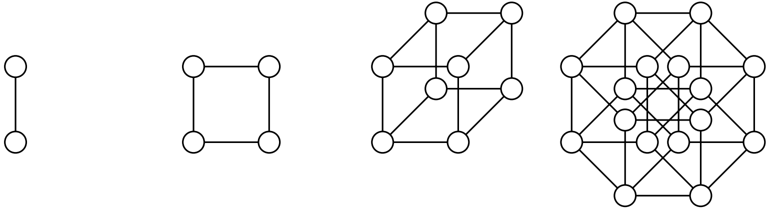
Introduction

Let G be a directed graph with possible loop edges on a vertex set V, so that G is just an arbitrary binary relation $G \subseteq V^2$.

A *domatic partition* is a partial function $f : V \rightharpoonup C$ which colors every out-neighborhood set $N_G(v) = \{w \in V : (v, w) \in G\}$ with all colors in C, i.e. $f[N_G(v)] = C$ for every $v \in V$.

Example: Q_n, the loop-free undirected n-regular hypercube graphs. Does Q_n admit domatic n-partitions?



Q_1 Q_2 Q_3 Q_4

Introduction

Theorem (Zelinka 1982): Q_n admits a domatic n-partition if and only if n is a power of two.

$Q_{\aleph_0}$ is the graph on $V = 2^{\mathbb{N}}$, where $(v, w) \in Q_{\aleph_0}$ whenever $v, w : \mathbb{N} \to \{0, 1\}$ differ at exactly one place.

$Q_{\aleph_0}$ is a *Schreier graph*:

Definition

Let $\Gamma \curvearrowright X$ be a group action, and let $S \subseteq \Gamma$ be a subset. Define:

$$\mathrm{Sch}(\Gamma, S, X) = \{(x, s \cdot x) \in X^2 : x \in X, s \in S\}$$

Let $\Gamma = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$, let $S = \{s \in \Gamma : s(i) = 1 \text{ exactly once}\}$, and let $\Gamma \curvearrowright \Gamma$ by left multiplication. Then $Q_{\aleph_0} \cong \mathrm{Sch}(\Gamma, S, \Gamma)$.

Introduction

Theorem (H)

Let Γ be a compact Polish group of finite Lebesgue covering dimension. Let $S \subseteq \Gamma$. Let $\Gamma \curvearrowright \Gamma$ by left multiplication. Let $G = \mathrm{Sch}(\Gamma, S, \Gamma)$.

1. *The following are equivalent:*
   - *G admits a domatic $\aleph_0$-partition with open parts.*
   - *G admits a domatic $\aleph_0$-partition with Baire measurable parts.*
   - *The topological closure $\overline{S} \subseteq \Gamma$ is uncountable.*
2. *Let μ be any Borel probability measure on Γ. If $S \subseteq \Gamma$ is infinite, then G admits a μ-measurable domatic $\aleph_0$-partition.*

$Q_{\aleph_0}$ does not admit domatic $\aleph_0$-partitions with Borel or Baire measurable parts, but $Q_{\aleph_0}$ admits a domatic $\aleph_0$-partition with Haar_Γ measurable parts.

Helpful facts

Let $G \subseteq X^2$. Note that if $1 \leq k \leq \aleph_0$, and $\mathcal{F}$ is a σ-algebra on X, then G admits an $\mathcal{F}$-measurable *partial* domatic k-partition if and only if G admits an $\mathcal{F}$-measurable *total* domatic k-partition.

Let $G \subseteq H \subseteq X^2$. If $f : X \rightharpoonup C$ is a domatic partition for G, then f is also a domatic partition for H.

Let $G \subseteq X^2$, let $f : X \rightharpoonup C$ be a domatic $|C|$-partition, and let $C' \subset C$. The restriction $g = f \restriction f^{-1}[C'] : X \rightharpoonup C'$ is also a domatic $|C'|$-partition, where $|C'| \leq |C|$.

Helpful facts

In the remainder of this talk, Γ will always be a Polish group.

If Γ acts on X and X is a Polish space, then we assume the action is continuous. If X is a Borel space, then we assume the action is Borel.

If Γ acts on X, we let $E_\Gamma^X \subseteq X^2$ be its orbit equivalence relation.

A *Borel graph* is a Borel subset $G \subseteq X^2$.
We let $E_G \subseteq X^2$ be its connectedness equivalence relation.

$\aleph_0$-partitions on $G = \mathrm{Sch}(\Gamma, S, \Gamma)$: Part 1

Theorem

Let $\Gamma \curvearrowright X$ be continuous, and let $S \subseteq \Gamma$ be countable compact. Then $G = \mathrm{Sch}(\Gamma, S, X)$ has no Baire measurable domatic $\aleph_0$-partition $f : X \to \mathbb{N}$.

Proof.

1. Every Baire measurable $f : X \to \mathbb{N}$ is continuous on an $E^X_{\langle S \rangle}$-invariant comeager set $A \subseteq X$.
2. If $a \in A$, then by continuity, $f[S \cdot a] \subseteq \mathbb{N}$ is a compact set, i.e. finite in $\mathbb{N}$.
3. Therefore $f[N_G(a)] = f[S \cdot a] \neq \mathbb{N}$, comeagerly in X. □

This also holds whenever $\overline{S} \supseteq S$ is countable compact.

The Lovász local lemma

Lemma (Lovász)

Let $A_0, A_1, \ldots, A_{n-1}$ be events in a probability space. Assume that:

- *Every event A_i is dependent on at most d other events.*
- *For all $i < n$, $\Pr[A_i] \leq p$.*
- *$ep(d+1) \leq 1$.*

Then $\bigwedge_{i<n} \overline{A_i}$ happens with nonzero probability.

Let $k \in \mathbb{N}$ be fixed and let $k \ll N \in \mathbb{N}$ be sufficiently large.

Let $G \subseteq X^2$ be a simple undirected N-regular finite graph.

Then a random k-partition $f : X \to \{0, \ldots, k-1\}$ is domatic with nonzero probability.

$\aleph_0$-partitions on $G = \mathrm{Sch}(\Gamma, S, \Gamma)$: Part 2

Theorem

Let Γ be a finite-dimensional compact Polish group, and assume $\overline{S} \subseteq \Gamma$ is uncountable. Then $G = \mathrm{Sch}(\Gamma, S, \Gamma)$ has a domatic partition with $\aleph_0$ disjoint open parts.

We prove this theorem using a lemma on *finite* partitions:

Lemma (Key lemma)

Let Γ be as above. For every $k, n \in \mathbb{N}$, there exists an $N \in \mathbb{N}$, such that for any size-N finite sets $F_0, \ldots, F_{n-1} \subseteq \Gamma$, there exist disjoint <u>open</u> sets $D_0, \ldots, D_{k-1}$, such that each right translate $F_i \cdot \gamma$ intersects every part D_j.

Lemma (Key lemma)

Let Γ be as above. For every $k, n \in \mathbb{N}$, there exists an $N \in \mathbb{N}$, such that for any size-N finite sets $F_0, \dots, F_{n-1} \subseteq \Gamma$, there exist disjoint open sets $D_0, \dots, D_{k-1}$, such that each right translate $F_i \cdot \gamma$ intersects every part D_j.

We prove this lemma in three steps:

- Let $d := \dim(\Gamma) < \infty$.

Step 1/3: Show that every "sufficiently fine" open cover $\mathscr{U}$ of Γ (every translate $F_i \cdot \gamma$ intersects every $U \in \mathscr{U}$ at most once) can be shrinked into an *open-cell packing* $\mathscr{R}$ of Γ, such that every $F_i \cdot \gamma$ intersects $\geq N - d$ many open cells $R \in \mathscr{R}$.

Let $N = |F_i|$ and $d = \dim(\Gamma)$.

Step 1/3: Show that every "sufficiently fine" open cover $\mathscr{U}$ of Γ (every translate $F_i \cdot \gamma$ intersects every $U \in \mathscr{U}$ at most once) can be shrinked into an *open-cell packing* $\mathscr{R}$ of Γ, such that every $F_i \cdot \gamma$ intersects $\geq N - d$ many open cells $R \in \mathscr{R}$.

Proof.

Construct $\mathscr{R}$ in a length-ω induction, making sure that

$$\dim(\Gamma) > \dim(F_i^{-1} \cdot \partial R_0) > \dim(F_i^{-1} \cdot \partial R_0 \cap F_i^{-1} \cdot \partial R_1) > \dots$$

for every distinct collection $R_0, R_1, \dots \in \mathscr{R}$.
If $F_i \cdot \gamma$ intersects $(\Gamma \setminus \bigcup \mathscr{R}) = \bigcup \partial \mathscr{R}$ for $\geq d + 1$ times, then since $\partial \mathscr{R}$ is fine,

$$\gamma \in F_i^{-1} \cdot \partial R_0 \cap \dots \cap F_i^{-1} \cdot \partial R_d$$

with $R_0, \dots, R_d \in \mathscr{R}$ distinct.
But $F_i^{-1} \cdot \partial R_0 \cap \dots \cap F_i^{-1} \cdot \partial R_d$ has to be $\varnothing$. □

Let $d = \dim(\Gamma)$.

Step 2/3: There is a constant $M_\Gamma < \infty$ depending only on d, and a "sufficiently fine" open cover $\mathscr{U}$ of Γ, such that every translate of every $U \in \mathscr{U}$ intersects $\leq M_\Gamma$ many other $V \in \mathscr{U}$.

Proof.

By a theorem of Gleason–Yamabe, Γ is an inverse ω-limit of d-dimensional Lie groups.
So $1_\Gamma \in \Gamma$ has a neighborhood basis of symmetric open sets $B \ni 1_\Gamma$ of *bounded doubling*: at most $(2+\varepsilon)^d$ translates of B pack into $B^2 \subseteq \Gamma$.
Let $\mathscr{U}$ be a minimal covering of Γ using translates of only one such $B \ni 1_\Gamma$. □

(In reality, we need something slightly stronger than stated above, but the proof is the same...)

Lemma (Key lemma)

Let Γ be as above. For every $k, n \in \mathbb{N}$, there exists an $N \in \mathbb{N}$, such that for any size-N finite sets $F_0, \ldots, F_{n-1} \subseteq \Gamma$, there exist disjoint <u>*open*</u> *sets $D_0, \ldots, D_{k-1}$, such that each right translate $F_i \cdot \gamma$ intersects every part D_j.*

<u>Step 3/3:</u> Fix a sufficiently fine open cover $\mathscr{U}$ of locally bounded growth, and fix an open-cell packing $\mathscr{R}$ that shrinks $\mathscr{U}$.

Show that using the Lovász local lemma, a random k-coloring of the parts in $\mathscr{R}$ is exactly as desired: The locally bounded growth condition of $\mathscr{U}$ feeds into the locality requirement of Lovász local lemma. □

<u>Note:</u> The key lemma also holds if every F_i is infinite.

Theorem

Let Γ be a finite-dimensional compact Polish group, and assume $\overline{S} \subseteq \Gamma$ is uncountable. Then $G = \mathrm{Sch}(\Gamma, S, \Gamma)$ has a domatic partition with $\aleph_0$ disjoint open parts.

Let $D, P \subseteq \Gamma$. We say that D *dominates* P if every translate $P \cdot \gamma$ intersects D, i.e. $P \cdot \gamma \cap D \neq \varnothing$ for every $\gamma \in \Gamma$.

Lemma (Splitting lemma)

Let Γ be as above. If $U \subseteq \Gamma$ is open and dominates a nonempty perfect set $P \subseteq \Gamma$, then there are disjoint open sets $A_0, A_1 \subseteq U$, each of which dominates P.

Let $D, P \subseteq \Gamma$. We say that D *dominates* P if every translate $P \cdot \gamma$ intersects D, i.e. $P \cdot \gamma \cap D \neq \varnothing$ for every $\gamma \in \Gamma$.

Lemma (Splitting lemma)

Let Γ be as above. If $U \subseteq \Gamma$ is open and dominates a nonempty perfect set $P \subseteq \Gamma$, then there are disjoint open sets $A_0, A_1 \subseteq U$, each of which dominates P.

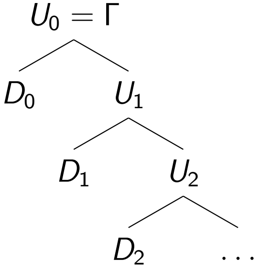
Proof.

Using compactness of Γ, find perfect subsets $P_0, \ldots, P_{n-1} \subseteq P$, such that for every $\gamma \in \Gamma$, there exists $i < n$, such that $P_i \cdot \gamma \subseteq U$. Apply the key lemma to $P_0, \ldots, P_{n-1}$ to get dominating sets $D_0, D_1 \subseteq \Gamma$, and let $A_j := U \cap D_j$. □

Theorem

Let Γ be a finite-dimensional compact Polish group, and assume $\overline{S} \subseteq \Gamma$ is uncountable. Then $G = \mathrm{Sch}(\Gamma, S, \Gamma)$ has a domatic partition with $\aleph_0$ disjoint open parts.

Fix a perfect set $P \subseteq \overline{S}$. By the splitting lemma:



$\langle D_n : n < \omega \rangle$ are disjoint and dominate P. Since each D_n is open, D_n necessarily also dominates S.
So $\langle D_n : n < \omega \rangle$ is an open domatic $\aleph_0$-partition for G. □

An application

Note that if we assume $S = \overline{S}$ is closed, then we can perform a full-binary-tree splitting of $U_0 = \Gamma$: In the splitting lemma, compactness implies we may assume $\overline{A_0}, \overline{A_1} \subseteq U$.

Corollary

Let Γ be as before. Let $P \subseteq \Gamma$ be a nonempty closed perfect set. Then there are $2^{\aleph_0}$ disjoint closed sets $\langle D_x : x \in 2^{\mathbb{N}} \rangle$, each of which dominates P.

Theorem (Erdős–Kunen–Mauldin 1981, rephrased)

Let $P \subseteq \mathbb{R}$ be a nonempty closed perfect set. Then there exists a Lebesgue null closed set D which dominates P.

$\mathbb{R}$ is not compact, but using $\Gamma = \mathbb{R}/\mathbb{Z}$ proves the above theorem.

An application

Let $D, P \subseteq \Gamma$. We say that D *dominates* P if every translate $P \cdot \gamma$ intersects D, i.e. $P \cdot \gamma \cap D \neq \varnothing$ for every $\gamma \in \Gamma$.

One can check that D dominates P if and only if $P^{-1} \cdot D = \Gamma$.

Theorem (Erdős–Kunen–Mauldin 1981)

Let $P \subseteq \mathbb{R}$ be a nonempty closed perfect set. Then there exists a Lebesgue null closed set D such that $P + D = \mathbb{R}$.

Theorem (H)

Let $1 \leq n \in \mathbb{N}$, and let $P \subseteq \mathbb{R}^n$ be a nonempty closed perfect set. Then there are $2^{\aleph_0}$ disjoint closed sets $\langle C_i : i < 2^{\aleph_0} \rangle$, such that $P + C_i = \mathbb{R}^n$ and $C_i + C_j = \mathbb{R}^n$ for all $i,j < 2^{\aleph_0}$ possibly equal.

The proof is very similar to the one we just saw.

Question

In the main theorem, can we drop the "finite dimension" assumption?

Unknown, even for $\Gamma = (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$...

$\aleph_0$-partitions on $G = \mathrm{Sch}(\Gamma, S, \Gamma)$: Part 3

Theorem (H)

Let $G \subseteq X^2$ be an out-degree $\aleph_0$-regular Borel graph with countable in-degrees on a Borel probability space (X, μ) of vertices.
Then G admits a μ-measurable domatic $\aleph_0$-partition.

Lemma (Recoloring lemma)

Let $G \subseteq X^2$ be as above. Assume there exists a Borel function $f : X \to \mathbb{N}$ such that every out-neighborhood is colored infinitely by f, i.e. every $f[N_G(x)] \subseteq \mathbb{N}$ is infinite.
Then G admits both μ-measurable and Baire measurable domatic $\aleph_0$-partitions.

Proof. Use a recoloring $g = (r \circ f) : X \to \mathbb{N}$, where $r \in \mathbb{N}^{\mathbb{N}}$ is picked using Fubini or Kuratowski–Ulam. □

Theorem (H)

Let $G \subseteq X^2$ be an out-degree $\aleph_0$-regular Borel graph with countable in-degrees on a Borel probability space (X, μ) of vertices.
Then G admits a μ-measurable domatic $\aleph_0$-partition.

Proof.

For every $k \in \mathbb{N}$ and $\varepsilon > 0$, a random coloring $f : X \to \{0, \ldots, k-1\}$ (defined over a finite Borel partition of X) can be shown to satisfy $f[N_G(x)] = \{0, \ldots, k-1\}$ over a μ-measure $\geq 1 - \varepsilon$ set of $x \in X$. Take a measure-theoretic limit as $(k, \varepsilon) \to (\infty, 0^+)$ using the Borel–Cantelli lemma. We get an $f : X \to [\mathbb{N}]^{<\infty}$ where every $N_G(x)$ is colored infinitely by f. Use the recoloring lemma to finish. □

It follows that if $S \subseteq \Gamma$ is infinite, then $G = \mathrm{Sch}(\Gamma, S, \Gamma)$ admits μ-measurable domatic $\aleph_0$-partitions.

Other results

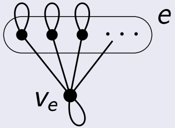
Theorem

There exists a fully looped, undirected, $\aleph_0$-regular, acyclic Borel graph $G \subseteq X^2$ on a Polish space X of vertices, without Baire measurable domatic 3-partitions.

Every such G admits Borel domatic 2-partitions, using a maximal independent set on a (Borel $\aleph_0$-colorable) function subgraph $G_f \subseteq G$.

Proof.

(Lecomte 2009) There exists an $\aleph_0$-uniform hypergraph version of the Kechris–Solecki–Todorcevic graph G_0.



For each hyperedge e, place a new vertex v_e which connects to all of e.
Check that this new ordinary graph G works. □

Edge-partitions

Let $G \subseteq X^2$ be a loop-free simple undirected $\aleph_0$-regular Borel graph.

A symmetric function on the edges $f : G \to C$ is *domatic* if it colors every edge-neighborhood with all colors in C, i.e. $f[\{x\} \times N_G(x)] = C$ for every $x \in X$.

Theorem

Let $G \subseteq X^2$ be as above. Then on an E_G-invariant conull or comeager set of vertices, G admits symmetric Borel domatic edge-$\aleph_0$-partitions.

Proof. Start with a Feldman–Moore edge-coloring, and use the recoloring lemma. □

Edge-partitions

Let $G \subseteq X^2$ be a loop-free simple undirected $\aleph_0$-regular Borel graph.

A symmetric function on the edges $f : G \to C$ is *domatic* if it colors every edge-neighborhood with all colors in C, i.e. $f[\{x\} \times N_G(x)] = C$ for every $x \in X$.

Theorem (Weilacher)

There exists a graph $G \subseteq X^2$ as above, which is moreover acyclic and Borel bipartite, such that G admits no Borel domatic edge-2-partitions and no Borel sinkless orientations.

The proof uses Marks' Borel determinacy graph technique.

Finite partitions on $G = \mathrm{Sch}(\Gamma, S, \Gamma)$

Theorem (Bernshteyn): A version of the Lovász local lemma holds in the measure and category settings.

Theorem (Csóka–Grabowski–Máthé–Pikhurko–Tyros): A version of the Lovász local lemma holds in the Borel and uniform subexponential growth setting.

Corollary

Let $\Gamma \curvearrowright X$ be a free Borel action, and let $S \subseteq \Gamma$ be infinite. Then:

1. $G = \mathrm{Sch}(\Gamma, S, X)$ *has measure-theoretic and Baire measurable domatic k-partitions for all $k \in \mathbb{N}$.*
2. *If every finitely generated subgroup of Γ has subexponential growth, then G has Borel domatic k-partitions for all $k \in \mathbb{N}$.*

Question: Is there a graph $G = \mathrm{Sch}(\Gamma, S, X)$ as above, without a Borel domatic k-partition for some $k \in \mathbb{N}$?

Thank you!