Measure equivalence of Baumslag-Solitar groups & type III relations

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Definition (Baumslag-Solitar groups)

Let $p, q \in \mathbf{Z} - \{0\}$. The **Baumslag-Solitar group** with parameters (p, q) is:

$$\mathrm{BS}(p,q) = \langle a, t \mid ta^p t^{-1} = a^q \rangle$$

Introduced in '62 by G. Baumslag et D. Solitar. First example of a 2-gen. 1-rel. non-Hopfian group.

> *[These groups] destroys several conjectures long and strongly held by [me] and others, and some published and unpublished "theorems".* *–B.H. Neumann*

- $\mathrm{BS}(p,q)\,\mathrm{isom}\,\mathrm{BS}(q,p)$
- $\mathrm{BS}(p,q)\,\mathrm{isom}\,\mathrm{BS}(-p,-q)$
- Moldavanskii '91: These are all the isomorphisms.
- $\mathrm{BS}(p,q)\,\mathrm{comm}\,\mathrm{BS}(-p,q)$, i.e they have isomorphic finite index subgroup.

Restrict to the case $1 \leq p \leq q$.

Three families of Baumslag-Solitar groups:

- $\mathrm{BS}(1, p)$ is solvable, amenable.
- $\mathrm{BS}(p, p)$ contains finite index $\mathbf{F} \times \mathbf{Z}$, where $\mathbf{F}$ is f.g free group.
- If $1 < p < q$. More heterogenous class.

Let Γ, Δ be countable groups.

Definition (Measure equivalence)

Γ and Δ are **measure equivalent**, denoted $\Gamma \,\mathrm{ME}\, \Delta$, if there exists actions $\Gamma, \Delta \curvearrowright (\Omega, \mu)$ on a standard measure space (e.g $\mathbf{N}, \mathbf{R}, 2^{\mathbf{N}}$... with Borel measures) such that:

- Actions are pmp, free and commute.
- Both actions admit (standard) fundamental domains of finite measure:

$$\mu\left(\Gamma \backslash \Omega\right), \mu\left(\Delta \backslash \Omega\right) < \infty$$

Introduced by Gromov in '93 as measure analogued of quasi-isometry. Examples:

- If $\Gamma < \Delta$ finite index subgroup, then $\Gamma \,\mathrm{ME}\, \Delta$.
- If $\Gamma \,\mathrm{comm}\, \Delta$, then $\Gamma \,\mathrm{ME}\, \Delta$.
- If Γ, Δ are both lattices in the same lcsc group G, then $\Gamma \,\mathrm{ME}\, \Delta$.

Measure equivalence, restricted to the family of Baumslag-Solitar groups:

- Ornstein-Weiss '80 implies $\mathrm{BS}(1, p)$ forms a ME-class.
- Commensurability implies all $\mathrm{BS}(p, p)$ are ME.
- Kida '14 proves $\mathrm{BS}(p, p)$ actually form a ME-class.
- Partial results for the third family.

Theorem (D. Gaboriau, P., A. Tserunyan, R. Tucker-Drob, K. Wróbel, '24+)

For $1 < p < q$ and $1 < r < s$, we have the measure equivalence:

$$\mathrm{BS}(p, q)\,\mathrm{ME}\,\mathrm{BS}(r, s).$$

Completes the classification of $\mathrm{BS}(p, q)$ up to ME.

(X, μ) standard probability space e.g $[0, 1]$ with Lebesgue measure.

$E \subset X^2$ Borel equivalence relation with countable classes.

E preserves the measure class (**pcm**) of μ: if A is null, the saturation $[A]_E$ is null.

All such equivalence relations arises as non-singular action of countable groups (Feldman – Moore '77)

(X, μ, E) pcm: Radon-Nikodym cocycle $\mathfrak{w} : E \to (0, \infty)$ satisfying:

- For almost all $x \in X$, for all $y, z \in [x]_E$:
  $\mathfrak{w}(z, y)\mathfrak{w}(y, x) = \mathfrak{w}(z, x)$
- For all $f : E \to [0, \infty]$:

$$\int_X \sum_{y \in [x]_E} f(x, y) \,\mathrm{d}\mu = \int_X \sum_{y \in [x]_E} f(y, x)\mathfrak{w}(y, x) \,\mathrm{d}\mu$$

Denote $T_{p,q}$ for the transitive directed tree with indeg $\equiv p$ and outdeg $\equiv q$

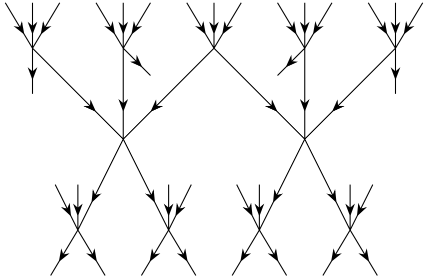

Figure: $T_{2,3}$

Definition (Cocycle $T_{p,q}$ treeing)

A **cocycle** $T_{p,q}$ **treeing** on (X, μ) is a Borel directed graph $G \subset X^2$ such that:

- Each connected component is isomorphic to $T_{p,q}$
- For each directed edge $y \to x$,

$$\mathfrak{w}(y, x) = \frac{p}{q}$$

Abbreviate to $\mathbf{CT}_{p,q}$.

$CT_{p,q}$ on $(X, \mu) \leftrightsquigarrow$ free pmp $\mathrm{Aut}(T_{p,q}) \curvearrowright (\widetilde{X}, \widetilde{\mu})$

$\mathrm{Aut}(T_{p,q})\,\mathrm{ME}\,\mathrm{Aut}(T_{r,s}) \Longrightarrow \mathrm{BS}(p,q)\,\mathrm{ME}\,\mathrm{BS}(r,s)$

Question

Given (X, ν, G) a $CT_{r,s}$, does there exist a prob. measure $\mu \sim \nu$ and a graph G' with the same conn. comp. s.t (X, μ, G') is a $CT_{p,q}$?

Cost and related notions give obstructions to similar questions.

Crucial definition:

Definition

(X, μ, E) pcm is **type III** if there are no σ-finite measures $\nu \sim \mu$ preserved by E, aka $\mathfrak{w}_\mu \equiv 1$.

All $CT_{p,q}$ are type III.

Theorem (P., '24+)

If (X, μ, E) is pcm, type III, aperiodic, treeable and $d \in \{2, ..., \infty\}$, then there is a free mcp action $\mathbf{F}_d \curvearrowright (X, \mu)$ such that $E = E_{\mathbf{F}_d}$.

Techniques to prove this theorem are useful to construct $CT_{p,q}$.

Let (X, E, μ) be pcm.

Definition

The **Maharam extension** of E on $X \times \mathbf{R}$ is defined by

$$(x, r)\ E^M\ (y, s) \Longleftrightarrow x\ E\ y\ \&\ r - \log \mathfrak{w}(y, x) = s$$

The Maharam E^M preserves the measure $\mu \times e^t \mathrm{d}t$.

The flow $\mathbf{R} \curvearrowright X \times \mathbf{R}$ on the second coordinate preserves E^M.

Let (X, E, μ) be pcm and E^M its Maharam extension on $X \times \mathbf{R}$.

Definition

The **Krieger flow** $\mathbf{K}_E$ is the induced mcp flow $\mathbf{R} \curvearrowright \Omega$ on the space of ergodic components of E^M.

For our proof, Kronecker flows on torii $\mathbf{R} \curvearrowright [0, \log \frac{q}{p}) \times [0, \log \frac{s}{r})$.

Krieger flow of a pmp action of $\mathrm{Aut}(T_{p,q}) \curvearrowright U \times [0, \log \frac{s}{r})$, where U is mixing and act by modular homomorphism on second coordinate.

If $F \leq E$ is a subequivalence relation, there is mcp factor map $\mathbf{K}_F \twoheadrightarrow \mathbf{K}_E$.

Theorem (D. Gaboriau, P., A. Tserunyan, R. Tucker-Drob, K. Wróbel, '24+)

Let (X, μ, G) be a type III mcp graph with aperiodic Krieger flow $\mathbf{K}_E$. Then there is a subgraph $H \leq G$ for which the factor $\mathbf{K}_H \twoheadrightarrow \mathbf{K}_G$ is a conjugacy.

Question: What about periodic Krieger flow?

Thank you!

Extra info

Let (X, E, μ) be a CBER. An ergodic decomposition is a, E-invariant map $\pi : X \to Z$ with the property that every E-invariant map $\rho : X \to Y$ factors through π: there is $f : Z \to Y$ such that $\rho = f \circ \pi$.