Hyperhyperfiniteness and complexity

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Countable Borel equivalence relations

A **countable Borel equivalence relation (CBER)** is a Borel equivalence relation on a standard Borel space X with every class countable.

Example

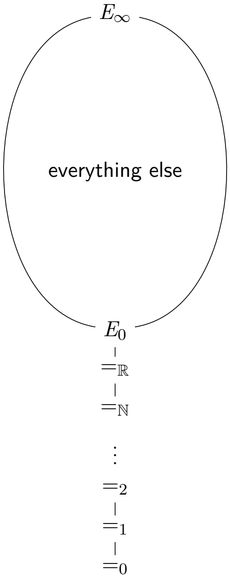
A Borel action $\Gamma \curvearrowright X$ of a countable group generates a CBER.

- $\mathbb{Z} \curvearrowright S^1$
- $\Gamma \curvearrowright 2^\Gamma$
- $F_2 \curvearrowright 2^{F_2}$

Borel reduction preorder on CBERs:
$E \leq_B F$ iff there is a Borel map $f\colon X \to Y$ with

$$x \, E \, x' \iff f(x) \, F \, f(x')$$

The CBERs under $\leq_B$



Hyperfinite CBERs

finite CBER: every equivalence class is finite.
hyperfinite CBER: increasing union of finite CBERs.
E_0 on $2^{\mathbb{N}}$:

$$x \, E_0 \, y \iff x_n = y_n \text{ for cofinitely many } n$$

E_0 is induced by the locally finite group $(\mathbb{Z}/2)^{<\mathbb{N}} \curvearrowright 2^{\mathbb{N}}$.
Non-example:

The CBER induced by $F_2 \curvearrowright 2^{F_2}$ is NOT hf.

(because F_2 is non-amenable).

Theorem (Dougherty-Jackson-Kechris)

E is hyperfinite iff $E \leq_B E_0$.

Hyperfinite = nontrivial but barely

Beyond hyperfinite I: hyper hyper hyper hyper hyper

Question (Union problem)

Are the hyperfinite CBERs closed under increasing union?

If not, then we get a new notion:

$$\text{hyperhyperfinite} := \text{increasing union of hyperfinite}$$

We can keep going:

- hyperfinite
- hyperhyperfinite
- hyperhyperhyperfinite
- ...
- hyper^k-finite ($k \in \mathbb{N}$)
- hyper$^\alpha$-finite ($\alpha < \omega_1$)

Beyond hyperfinite II: scattered orders

Theorem (Slaman-Steel, Weiss)

A CBER E is hyperfinite iff every class can be $\mathbb{Z}$-ordered (in a Borel way).

A linear order is **scattered** if it doesn't contain a $\mathbb{Q}$.
Linear orders have **Hausdorff derivative** $L \mapsto L'$ (glues adjacent points).

$$L \text{ is scattered} \iff L^{(\infty)} = 1$$

$\mathrm{rank}(L)$ is smallest α with $L^{(\alpha)} = 1$.
$\mathrm{rank}(\mathbb{Z}) = 1$.
Scattered order hierarchy:

- Hyperfinite = "orderable by rank 1"
- "orderable by rank 2"
- $\ldots$
- "orderable by rank α" ($\alpha < \omega_1$)

Beyond hyperfinite III exclusive preview: Amenability

Hyperfinite $\implies$ **amenable**.
Reiter function on E:

assigns to each $x \in X$ a prob measure p^x on $[x]_E$.

E is **amenable**:

there is a sequence $(p_n)_n$ of Reiter functions such that
for all $(x, y) \in E$,
$\lim_n \|p_n^x - p_n^y\| = 0$.

Definition of amenable group via Reiter functions gives

Γ is amenable $\implies$ CBER induced by $\Gamma \curvearrowright X$ is amenable

Question (Weiss's question)

If Γ is amenable, is every $\Gamma \curvearrowright X$ hyperfinite?

Beyond hyperfinite III: α-amenability

amenable: $\forall^\infty n \, \|p_n - q_n\| < \varepsilon$.
2-amenable: $\forall^\infty n \, \forall^\infty m \, \|p_{n,m} - q_{n,m}\| < \varepsilon$
α-amenable: converges with respect to Fin^α ideal

there is a Reiter sequence $(p_n)_n$ such that
for all $(x, y) \in E$,
$$\mathrm{Fin}^\alpha \lim_n \|p_n^x - p_n^y\| = 0.$$

Amenability hierarchy:

- amenable
- 2-amenable
- ...
- α-amenable ($\alpha < \omega_1$)

hyper$^\alpha$-finite $\implies$ α-amenable

orderable by rank α $\implies$ α-amenable

Measure

E: CBER on X
For a Borel probability measure μ on X:

$$E \text{ is } \mu\text{-凸凹} \iff E\restriction Y \text{ is 凸凹 for some } \mu\text{-conull } Y$$

Examples: μ-hyperfinite, μ-treeable.

$$\text{measure-凸凹} \iff \mu\text{-凸凹 for all } \mu$$

Examples: measure-hyperfinite, measure-treeable.
All measure versions of hyperfinite are equivalent!

Measure-hyperfinite: The final frontier

Theorem

The following are equivalent:

1. *measure-hyperfinite*
2. *measure-α-amenable*
3. *measure-[orderable by rank α]*
4. *measure-[anything else that can reasonably be called amenable]*

The hardest result here is (2) implies (1), due to Connes-Feldman-Weiss.

Question

Does measure-hyperfinite imply hyperfinite?

A complexity obstruction?

Hyperfiniteness is $\mathbf{\Sigma}^1_2$ (the set of codes of hyperfinite CBERs is $\mathbf{\Sigma}^1_2$)

"There exists an increasing sequence of CBERs such that ..."

Measure-hyperfiniteness is $\mathbf{\Pi}^1_1$ (Segal).

$$\text{measure-hyperfinite is hyperfinite} \implies \text{hyperfinite is } \mathbf{\Pi}^1_1$$

A reasonable possibility in the other direction:

Hyperfiniteness is $\mathbf{\Sigma}^1_2$-complete.

The two extremes

Measure-hyperfinite = hyperfinite
(monotonous and dull)

- Hyperfinite is $\mathbf{\Pi}^1_1$.
- Union Problem has a positive answer.
- Weiss's question has a positive answer.

Pure chaos
(vibrant and exciting)

- Hyperfinite is $\mathbf{\Sigma}^1_2$-complete.
- The hyper$^\alpha$-hierarchy is strict.
- The "orderable by $\operatorname{rank}\alpha$" hierarchy is strict.
- The α-amenable hierarchy is strict.
- No secret implications.

The theorem

General belief: We're close to one of the two extremes.
We give evidence supporting this.

Theorem (Frisch-信-Vidnyánszky)

If there is a hhf non-hf CBER, then hyperfiniteness is $\boldsymbol{\Sigma}^1_2$-complete.

Consequence of $\boldsymbol{\Sigma}^1_2$-completeness:
there is no dichotomy for hyperfiniteness.
In general:

$$\text{Dichotomy for 凸凹} \implies \text{凸凹 is } \boldsymbol{\Delta}^1_2$$

Smoothness is $\boldsymbol{\Sigma}^1_2$ (there is a Borel reduction from E to equality).
Non-smoothness is **also** $\boldsymbol{\Sigma}^1_2$ (there is a Borel reduction from E_0 to E).
So smoothness is $\boldsymbol{\Delta}^1_2$.

The proof: a scheme associated to a hhf CBER

Suppose E is the increasing union of hyperfinite E_n.
There are finite CBERs $(F_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ such that

1. $F_s \subseteq F_t$ for $s \preceq t$.
2. $F_{s^\frown 0} \subseteq F_{s^\frown 1} \subseteq F_{s^\frown 2} \subseteq \cdots$, with union $E_{|s|}$.

For $x \in \mathbb{N}^{\mathbb{N}}$, define $E_x = \bigcup_n F_{x \restriction n}$ (this is hyperfinite).
Main property: for every $e \in E$, almost every branch contains e:

$$\{x \in \mathbb{N}^{\mathbb{N}} : e \notin E_x\} \text{ is } K_\sigma.$$

Reason:

- e appears eventually, say in E_7.
- For every s with $|s| \geq 7$, only finitely many failures below s.
- So for every $s \in \mathbb{N}^7$, tree of failures below s is finitely branching, i.e. compact.

Main idea: hyperfinite is analytic-hard

Let E be hhf non-hf, and fix the scheme $(F_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$.
Given a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, define $E_T = \bigcap_{x\in[T]} E_x$.

$$T \text{ is ill-founded} \implies E_T \subseteq E_x \text{ is hf}$$
$$T \text{ is well-founded} \implies E_T = E \text{ is non-hf}$$

The E_T are the sections of

$$\{(T, e) \in \text{Trees} \times E : \forall x \in \mathbb{N}^{\mathbb{N}}(x \in [T] \implies e \in E_x)\}.$$

Is it Borel? Its complement is the projection of the Borel set

$$\{(T, e, x) \in \text{Trees} \times E \times \mathbb{N}^{\mathbb{N}} : x \in [T] \text{ and } e \notin E_x)\}.$$

Fiber above (T, e) is $\mathbf{\Pi}^0_1 \cap K_\sigma = K_\sigma$.
Done by K_σ-uniformization.

Upgrading to $\mathbf{\Sigma}^1_2$-hard

Recall for $x, y \in [\mathbb{N}]^{\mathbb{N}}$:

$$x \leq^* y \iff x(n) \leq y(n) \text{ for large enough } n.$$

The **cone above** y is

$$\{x \in [\mathbb{N}]^{\mathbb{N}} : x \geq^* y\}.$$

A **non-dominating set** is a subset of $[\mathbb{N}]^{\mathbb{N}}$ which is disjoint from a cone.
The previous proof basically does the following:

1. Start with a non-hf CBER E on $[\mathbb{N}]^{\mathbb{N}}$.
2. Show that E is hf on every non-dominating set (this uses hhf).
3. Conclude that hyperfiniteness is analytic-hard.

Actually, we showed hf **uniformly** on all non-dominating sets.
This pushes it to $\mathbf{\Sigma}^1_2$-hard.

Hyperfinite is $\mathbf{\Sigma}^1_2$-hard

Theorem

Suppose there is a non-hf CBER E on $[\mathbb{N}]^{\mathbb{N}}$ such that the restriction of $=_{[\mathbb{N}]^{\mathbb{N}}} \times E$ to $\{(x,y) \in [\mathbb{N}]^{\mathbb{N}} \times [\mathbb{N}]^{\mathbb{N}} : x \not\leq^* y\}$ is hf. Then hyperfiniteness is $\mathbf{\Sigma}^1_2$-complete.

This is actually a general fact about homomorphism problems.
(E is hf iff $(E, E^c) \to (E_0, E_0^c)$)

Theorem

*Let $\mathcal{H}$ be a Borel $\mathcal{L}$-structure.
Suppose there is a Borel $\mathcal{L}$-structure $\mathcal{G}$ on $[\mathbb{N}]^{\mathbb{N}}$ with no hom to $\mathcal{H}$, whose restriction to every non-dominating set uniformly homs to $\mathcal{H}$.
Then "having a hom to $\mathcal{H}$" is $\mathbf{\Sigma}^1_2$-complete.*

Essentially already in Todorčević-Vidnyánszky.
They showed 3-colorability is $\mathbf{\Sigma}^1_2$-complete (using $\mathcal{H} = K_3$).

Questions

"Hyperfinite is $\boldsymbol{\Sigma}^1_2$-complete" says nothing about the Union Problem.
A potential equivalence:

Question

Is "Hyperfinite is $\boldsymbol{\Sigma}^1_2$-complete" equivalent to "There is a measure-hf non-hf CBER"?

Unclear how to use **every** measure.
Another direction:
Non-dominating sets form a σ-ideal.
Try other σ-ideals?

Question

Is there a non-hyperfinite CBER on $\mathbb{N}^{\mathbb{N}}$ which is hyperfinite on every compact set?