Measurable Brooks's Theorem for Directed Graphs

Cecelia Higgins

University of California, Berkeley and University of California, Los Angeles

Caltech Logic Seminar 10/16/24

Brooks's Theorem

Let G be a finite undirected graph.

Brooks's Theorem

Let G be a finite undirected graph.

- A map $c : V(G) \to \{\text{colors}\}$ is a **(proper) coloring** of G if x, y adjacent $\implies c(x) \neq c(y)$ for all x, y.

Brooks's Theorem

Let G be a finite undirected graph.

- A map $c : V(G) \to \{\text{colors}\}$ is a **(proper) coloring** of G if x, y adjacent $\implies c(x) \neq c(y)$ for all x, y.
- The **chromatic number** $\chi(G)$ of G is the least number of colors needed to give a coloring of G.

Brooks's Theorem

Let G be a finite undirected graph.

- A map $c : V(G) \to \{\text{colors}\}$ is a **(proper) coloring** of G if x, y adjacent $\implies c(x) \neq c(y)$ for all x, y.
- The **chromatic number** $\chi(G)$ of G is the least number of colors needed to give a coloring of G.

Proposition (Greedy bound)

Suppose each vertex in G has degree at most d. Then $\chi(G) \leq d + 1$.

Brooks's Theorem

Proposition (Greedy bound)

Suppose each vertex in G has degree at most d. Then $\chi(G) \leq d + 1$.

Brooks's Theorem

Proposition (Greedy bound)

Suppose each vertex in G has degree at most d. Then $\chi(G) \leq d + 1$.

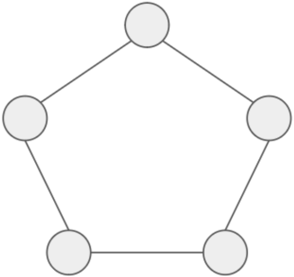
This bound is sharp.

Brooks's Theorem

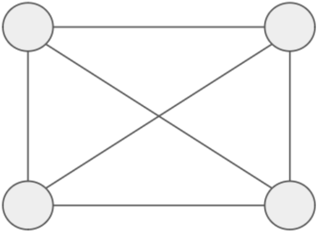
Proposition (Greedy bound)

Suppose each vertex in G has degree at most d. Then $\chi(G) \leq d + 1$.

This bound is sharp.



Odd cycles



Complete graphs

Brooks's Theorem

However, these are the only obstructions.

Brooks's Theorem

However, these are the only obstructions.

Theorem (Brooks's theorem)

Suppose each vertex of G has degree at most $d \geq 2$. If $d = 2$, suppose G has no odd cycles; if $d \geq 3$, suppose G does not contain the complete graph on $d + 1$ vertices. Then $\chi(G) \leq d$.

Failure of Borel Brooks's Theorem

Let G be an undirected **Borel graph**:

Failure of Borel Brooks's Theorem

Let G be an undirected **Borel graph**:

- $V(G)$ is a standard Borel space X.

Failure of Borel Brooks's Theorem

Let G be an undirected **Borel graph**:

- $V(G)$ is a standard Borel space X.
- $E(G)$ is Borel as a subset of X^2.

Failure of Borel Brooks's Theorem

Let G be an undirected **Borel graph**:

- $V(G)$ is a standard Borel space X.
- $E(G)$ is Borel as a subset of X^2.

The **Borel chromatic number** $\chi_B(G)$ of G is the least number of colors needed to give a Borel coloring of G.

Failure of Borel Brooks's Theorem

Let G be an undirected **Borel graph**:

- $V(G)$ is a standard Borel space X.
- $E(G)$ is Borel as a subset of X^2.

The **Borel chromatic number** $\chi_B(G)$ of G is the least number of colors needed to give a Borel coloring of G.

Fact

[Kechris–Solecki–Todorčević, 1999] There is a Borel graph G for which $\chi(G) = 2$ but $\chi_B(G)$ is uncountable.

Failure of Borel Brooks's Theorem

Is the following true?

Question: Borel greedy bound?

Suppose each vertex of G has degree at most d. Then is $\chi_B(G) \leq d + 1$?

Failure of Borel Brooks's Theorem

Is the following true?

Question: Borel greedy bound?

Suppose each vertex of G has degree at most d. Then is $\chi_B(G) \leq d + 1$?

Yes [Kechris–Solecki–Todorčević, 1999].

Failure of Borel Brooks's Theorem

Is the following true?

Question: Borel Brooks's theorem?

Suppose each vertex of G has degree at most $d \geq 2$ and G contains no odd cycles or complete graphs. Then is $\chi_B(G) \leq d$?

Failure of Borel Brooks's Theorem

Is the following true?

Question: Borel Brooks's theorem?

Suppose each vertex of G has degree at most $d \geq 2$ and G contains no odd cycles or complete graphs. Then is $\chi_B(G) \leq d$?

No – by a 2016 theorem of Marks, there are acyclic d-regular Borel graphs having Borel chromatic number equal to $d+1$.

Failure of Borel Brooks's Theorem

Is the following true?

Question: Borel Brooks's theorem?

Suppose each vertex of G has degree at most $d \geq 2$ and G contains no odd cycles or complete graphs. Then is $\chi_B(G) \leq d$?

No – by a 2016 theorem of Marks, there are acyclic d-regular Borel graphs having Borel chromatic number equal to $d + 1$.

In fact, the set of acyclic d-regular Borel graphs having Borel chromatic number at most d is $\mathbf{\Sigma}^1_2$-complete [Brandt–Chang–Grebík–Grunau–Rozhoň–Vidnyánszky, 2024].

μ-Measurable Brooks's Theorem

Let μ be a Borel probability measure on X. The μ-**measurable chromatic number** $\chi_\mu(G)$ of G is the least number of colors needed to give a μ-measurable coloring of G.

μ-Measurable Brooks's Theorem

Let μ be a Borel probability measure on X. The μ-**measurable chromatic number** $\chi_\mu(G)$ of G is the least number of colors needed to give a μ-measurable coloring of G.

If we relax "Borel" to "μ-measurable" in the statement of Brooks's theorem, then we get a true claim.

μ-Measurable Brooks's Theorem

Let μ be a Borel probability measure on X. The **μ-measurable chromatic number** $\chi_\mu(G)$ of G is the least number of colors needed to give a μ-measurable coloring of G.

If we relax "Borel" to "μ-measurable" in the statement of Brooks's theorem, then we get a true claim.

Theorem (μ-measurable Brooks's theorem)

[Conley–Marks–Tucker-Drob, 2016] Suppose each vertex of G has degree at most $d \geq 3$ and G does not contain the complete graph on $d+1$ vertices. Then $\chi_\mu(G) \leq d$.

μ-Measurable Brooks's Theorem

Let μ be a Borel probability measure on X. The μ-**measurable chromatic number** $\chi_\mu(G)$ of G is the least number of colors needed to give a μ-measurable coloring of G.

If we relax "Borel" to "μ-measurable" in the statement of Brooks's theorem, then we get a true claim.

Theorem (μ-measurable Brooks's theorem)

[Conley–Marks–Tucker-Drob, 2016] Suppose each vertex of G has degree at most $d \geq 3$ and G does not contain the complete graph on $d + 1$ vertices. Then $\chi_\mu(G) \leq d$.

When $d = 2$, we encounter an ergodic theoretic obstruction.

Baire-Measurable Brooks's Theorem

The **Baire-measurable chromatic number** $\chi_{BM}(G)$ of G is the least number of colors needed to give a Baire-measurable coloring of G.

Baire-Measurable Brooks's Theorem

The **Baire-measurable chromatic number** $\chi_{BM}(G)$ of G is the least number of colors needed to give a Baire-measurable coloring of G.

Theorem (Baire-measurable Brooks's theorem)

[Conley–Marks–Tucker-Drob, 2016] Suppose each vertex of G has degree at most $d \geq 3$ and G does not contain the complete graph on $d+1$ vertices. Then $\chi_{BM}(G) \leq d$.

Baire-Measurable Brooks's Theorem

The **Baire-measurable chromatic number** $\chi_{BM}(G)$ of G is the least number of colors needed to give a Baire-measurable coloring of G.

Theorem (Baire-measurable Brooks's theorem)

[Conley–Marks–Tucker-Drob, 2016] Suppose each vertex of G has degree at most $d \geq 3$ and G does not contain the complete graph on $d+1$ vertices. Then $\chi_{BM}(G) \leq d$.

Again, when $d = 2$, there is an ergodic-theoretic obstruction.

Directed Graphs and Dicolorings

Let D be a **directed graph** (or **digraph**). Then D is given by:

Directed Graphs and Dicolorings

Let D be a **directed graph** (or **digraph**). Then D is given by:

- A vertex set.

Directed Graphs and Dicolorings

Let D be a **directed graph** (or **digraph**). Then D is given by:

- A vertex set.
- A set of **arcs** (ordered pairs of vertices).

Directed Graphs and Dicolorings

Let D be a **directed graph** (or **digraph**). Then D is given by:

- A vertex set.
- A set of **arcs** (ordered pairs of vertices).

Then:

- A map $c : V(D) \to \{\text{colors}\}$ is a **dicoloring** of D if no directed cycle in D is c-monochromatic.

Directed Graphs and Dicolorings

Let D be a **directed graph** (or **digraph**). Then D is given by:

- A vertex set.
- A set of **arcs** (ordered pairs of vertices).

Then:

- A map $c : V(D) \to \{\text{colors}\}$ is a **dicoloring** of D if no directed cycle in D is c-monochromatic.
- The **dichromatic number** $\vec{\chi}(D)$ of D is the least number of colors needed to give a dicoloring.

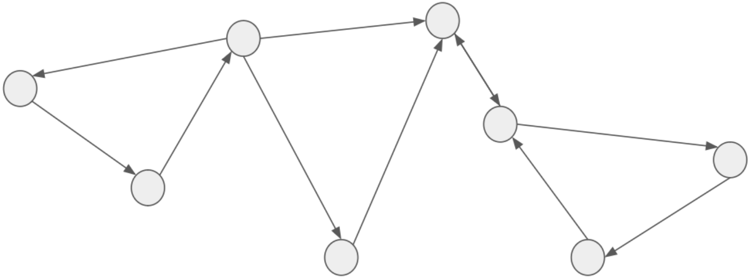
Directed Graphs and Dicolorings

Let D be a **directed graph** (or **digraph**). Then D is given by:

- A vertex set.
- A set of **arcs** (ordered pairs of vertices).

Then:

- A map $c : V(D) \to \{\text{colors}\}$ is a **dicoloring** of D if no directed cycle in D is c-monochromatic.
- The **dichromatic number** $\vec{\chi}(D)$ of D is the least number of colors needed to give a dicoloring.



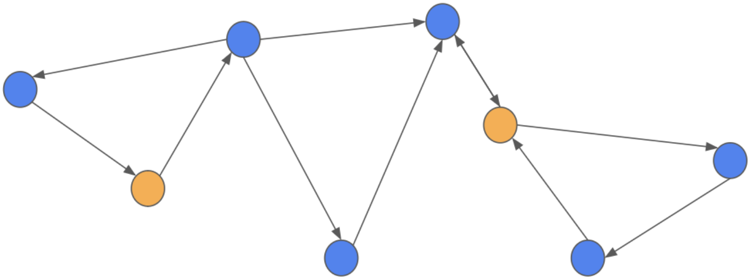
Directed Graphs and Dicolorings

Let D be a **directed graph** (or **digraph**). Then D is given by:

- A vertex set.
- A set of **arcs** (ordered pairs of vertices).

Then:

- A map $c : V(D) \to \{\text{colors}\}$ is a **dicoloring** of D if no directed cycle in D is c-monochromatic.
- The **dichromatic number** $\vec{\chi}(D)$ of D is the least number of colors needed to give a dicoloring.



Brooks's Theorem for Digraphs

Let D be a finite digraph.

Brooks's Theorem for Digraphs

Let D be a finite digraph.

Proposition (Greedy bound for digraphs)

Suppose the maximum degree of each vertex in D is at most d. Then $\overrightarrow{\chi}(D) \leq d + 1$.

Brooks's Theorem for Digraphs

Let D be a finite digraph.

Proposition (Greedy bound for digraphs)

Suppose the maximum degree of each vertex in D is at most d. Then $\vec{\chi}(D) \leq d + 1$.

Again, this bound is sharp, but because of just a few obstructions.

Brooks's Theorem for Digraphs

Let D be a finite digraph.

Proposition (Greedy bound for digraphs)

Suppose the maximum degree of each vertex in D is at most d. Then $\vec{\chi}(D) \leq d + 1$.

Again, this bound is sharp, but because of just a few obstructions.

Theorem (Brooks's theorem for digraphs)

[Harutyunyan–Mohar, 2011] Suppose the maximum degree of each vertex in D is at most $d \geq 2$. If $d = 2$, suppose D has no symmetrizations of odd cycles; if $d \geq 3$, suppose D does not contain the symmetrization of the complete graph on $d + 1$ vertices. Then $\vec{\chi}(D) \leq d$.

Failure of Borel Directed Brooks's Theorem

There is no Borel version of Brooks's theorem for digraphs;

Failure of Borel Directed Brooks's Theorem

There is no Borel version of Brooks's theorem for digraphs; take the symmetrization of one of Marks's undirected Borel graphs that is acyclic and d-regular but has no Borel d-coloring.

Failure of Borel Directed Brooks's Theorem

There is also a Borel digraph *without bidirected edges* witnessing the failure of Borel directed Brooks's theorem.

Failure of Borel Directed Brooks's Theorem

There is also a Borel digraph *without bidirected edges* witnessing the failure of Borel directed Brooks's theorem.

For each $i < 3$, let $\Gamma_i = \mathbb{Z}/4\mathbb{Z} = \langle \gamma_i | \gamma_i^4 = 1 \rangle$.

Failure of Borel Directed Brooks's Theorem

There is also a Borel digraph *without bidirected edges* witnessing the failure of Borel directed Brooks's theorem.

For each $i < 3$, let $\Gamma_i = \mathbb{Z}/4\mathbb{Z} = \langle \gamma_i | \gamma_i^4 = 1 \rangle$. Consider the left shift action of $\Gamma = \Gamma_0 * \Gamma_1 * \Gamma_2$ on $(2^{\mathbb{N}})^{\Gamma}$:

$$(\alpha \cdot x)(\beta) = x(\alpha^{-1}\beta).$$

Failure of Borel Directed Brooks's Theorem

There is also a Borel digraph *without bidirected edges* witnessing the failure of Borel directed Brooks's theorem.

For each $i < 3$, let $\Gamma_i = \mathbb{Z}/4\mathbb{Z} = \langle \gamma_i | \gamma_i^4 = 1 \rangle$. Consider the left shift action of $\Gamma = \Gamma_0 * \Gamma_1 * \Gamma_2$ on $(2^{\mathbb{N}})^{\Gamma}$:

$$(\alpha \cdot x)(\beta) = x(\alpha^{-1}\beta).$$

The directed Schreier graph of this action with generators $\gamma_0, \gamma_1, \gamma_2$ has no Borel 3-dicoloring.

Measurable Brooks's Theorem for Digraphs

There is a μ-measurable version (as well as a Baire-measurable version) of Brooks's theorem for digraphs.

Measurable Brooks's Theorem for Digraphs

There is a μ-measurable version (as well as a Baire-measurable version) of Brooks's theorem for digraphs.

Let D be a Borel digraph on a standard Borel space X with a Borel probability measure μ.

Measurable Brooks's Theorem for Digraphs

There is a μ-measurable version (as well as a Baire-measurable version) of Brooks's theorem for digraphs.

Let D be a Borel digraph on a standard Borel space X with a Borel probability measure μ.

Theorem (Measurable Brooks's theorem for digraphs)

[H., 2024] Suppose the maximum degree of each vertex in D is at most $d \geq 3$ and D does not contain the symmetrization of the complete graph on $d+1$ vertices. Then $\vec{\chi}_{\mu}(D) \leq d$.

Measurable Brooks's Theorem for Digraphs

Theorem (Measurable Brooks's theorem for digraphs)

[H., 2024] Suppose the maximum degree of each vertex in D is at most $d \geq 3$ and D does not contain the symmetrization of the complete graph on $d+1$ vertices. Then $\overrightarrow{\chi}_\mu(D) \leq d$.

Measurable Brooks's Theorem for Digraphs

Theorem (Measurable Brooks's theorem for digraphs)

[H., 2024] Suppose the maximum degree of each vertex in D is at most $d \geq 3$ and D does not contain the symmetrization of the complete graph on $d+1$ vertices. Then $\overrightarrow{\chi}_\mu(D) \leq d$.

Proof outline:

1. Use the one-ended function technique of Conley–Marks–Tucker-Drob to show that each component of D can be dicolored outside of a carefully chosen Borel set.

Measurable Brooks's Theorem for Digraphs

Theorem (Measurable Brooks's theorem for digraphs)

[H., 2024] Suppose the maximum degree of each vertex in D is at most $d \geq 3$ and D does not contain the symmetrization of the complete graph on $d+1$ vertices. Then $\overrightarrow{\chi}_\mu(D) \leq d$.

Proof outline:

1. Use the one-ended function technique of Conley–Marks–Tucker-Drob to show that each component of D can be dicolored outside of a carefully chosen Borel set.
2. Dicolor the "easy" components: Components having at least one vertex with minimum degree less than d, and components that are not Gallai trees.

Measurable Brooks's Theorem for Digraphs

Theorem (Measurable Brooks's theorem for digraphs)

[H., 2024] Suppose the maximum degree of each vertex in D is at most $d \geq 3$ and D does not contain the symmetrization of the complete graph on $d+1$ vertices. Then $\overrightarrow{\chi}_\mu(D) \leq d$.

Proof outline:

1. Use the one-ended function technique of Conley–Marks–Tucker-Drob to show that each component of D can be dicolored outside of a carefully chosen Borel set.
2. Dicolor the "easy" components: Components having at least one vertex with minimum degree less than d, and components that are not Gallai trees. These components can be dicolored *in a Borel way*.

Measurable Brooks's Theorem for Digraphs

Theorem (Measurable Brooks's theorem for digraphs)

[H., 2024] Suppose the maximum degree of each vertex in D is at most $d \geq 3$ and D does not contain the symmetrization of the complete graph on $d+1$ vertices. Then $\overrightarrow{\chi}_\mu(D) \leq d$.

Proof outline:

1. Use the one-ended function technique of Conley–Marks–Tucker-Drob to show that each component of D can be dicolored outside of a carefully chosen Borel set.
2. Dicolor the "easy" components: Components having at least one vertex with minimum degree less than d, and components that are not Gallai trees. These components can be dicolored *in a Borel way*.
3. Dicolor the "hard" components, the infinite Gallai trees, *by discarding a μ-null set*.

Gallai Trees

Let G be the underlying undirected graph of D.

Let G be the underlying undirected graph of D.

- A set $S \subseteq V(G)$ is **biconnected** if the subgraph $G[S]$ of G induced by S is connected and $G[S \setminus \{s\}]$ is connected for each $s \in S$.

Gallai Trees

Let G be the underlying undirected graph of D.

- A set $S \subseteq V(G)$ is **biconnected** if the subgraph $G[S]$ of G induced by S is connected and $G[S \setminus \{s\}]$ is connected for each $s \in S$.
- A **block** in G is a maximal biconnected subset of $V(G)$.

Let G be the underlying undirected graph of D.

- A set $S \subseteq V(G)$ is **biconnected** if the subgraph $G[S]$ of G induced by S is connected and $G[S \setminus \{s\}]$ is connected for each $s \in S$.
- A **block** in G is a maximal biconnected subset of $V(G)$.
- A block S in G is **bad** if the sub*digraph* $D[S]$ of D induced by S is a directed cycle, the symmetrization of an odd cycle, or the symmetrization of a complete graph.

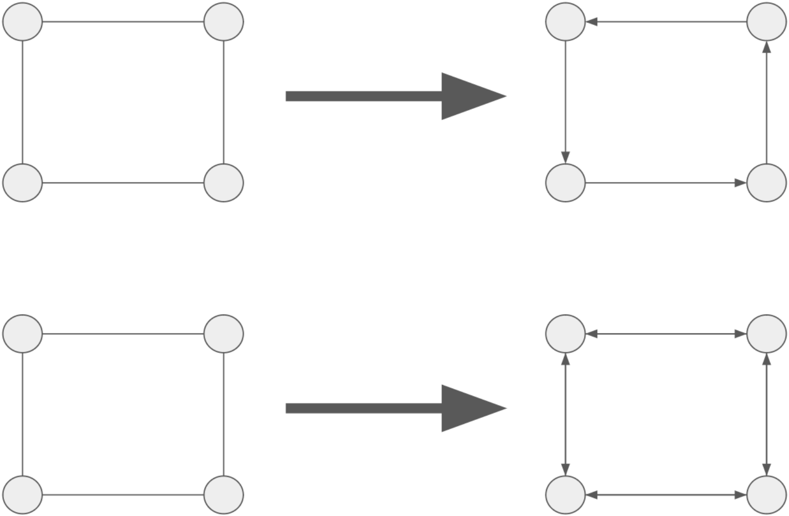
Gallai Trees

Let G be the underlying undirected graph of D.

- A set $S \subseteq V(G)$ is **biconnected** if the subgraph $G[S]$ of G induced by S is connected and $G[S \setminus \{s\}]$ is connected for each $s \in S$.
- A **block** in G is a maximal biconnected subset of $V(G)$.
- A block S in G is **bad** if the sub*digraph* $D[S]$ of D induced by S is a directed cycle, the symmetrization of an odd cycle, or the symmetrization of a complete graph.
- A connected digraph is a **Gallai tree** if all its blocks are bad.

Gallai Trees

Underlying graph Directed graph

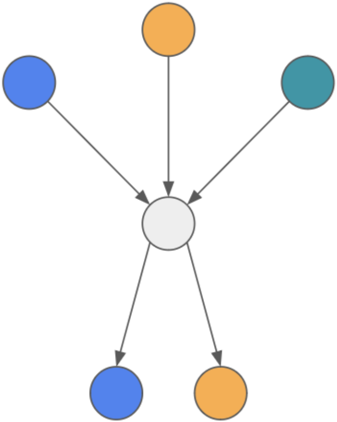


“Easy” Components

Components having at least one vertex with minimum degree less than d:

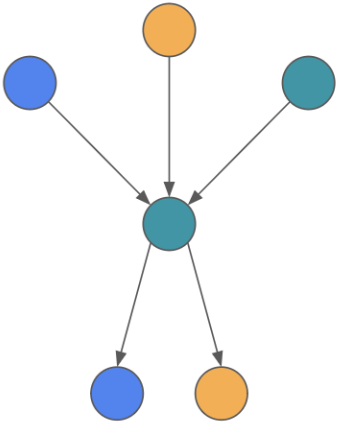
"Easy" Components

Components having at least one vertex with minimum degree less than d:



“Easy” Components

Components having at least one vertex with minimum degree less than d:

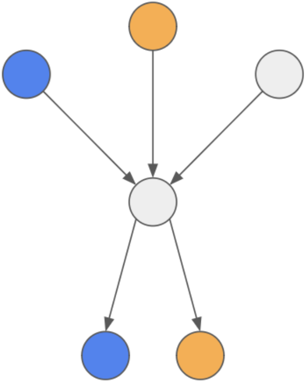


“Easy” Components

How do we ensure that no two reserved vertices are adjacent?

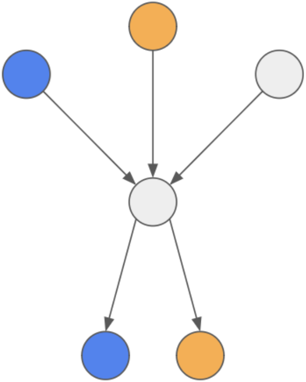
“Easy” Components

How do we ensure that no two reserved vertices are adjacent?



“Easy” Components

How do we ensure that no two reserved vertices are adjacent?



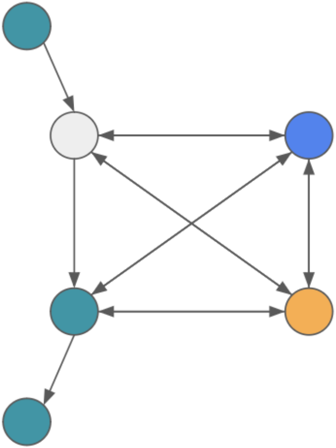
Select the reserved vertices from an independent set of vertices with minimum degree less than d.

“Easy” Components

Components that are not Gallai trees:

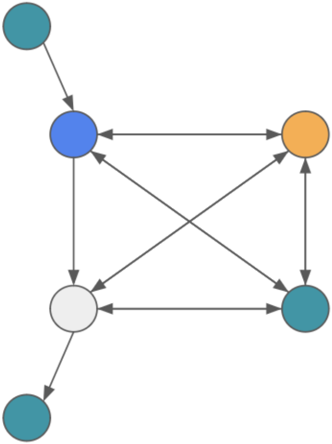
“Easy” Components

Components that are not Gallai trees:



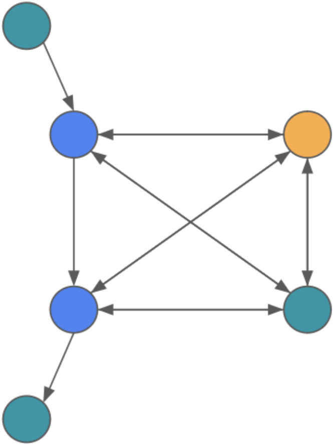
"Easy" Components

Components that are not Gallai trees:



“Easy” Components

Components that are not Gallai trees:



“Easy” Components

How do we ensure that color shifts on different blocks do not interfere with one another?

“Easy” Components

How do we ensure that color shifts on different blocks do not interfere with one another?

Select the reserved good blocks so that, for distinct selected blocks S, T, no neighbor of S is also a neighbor of T.

Johansson's Theorem for Digraphs?

For an undirected graph G, a result known as Johansson's theorem improves on the upper bound on the chromatic number from Brooks's theorem when G is triangle-free.

Johansson's Theorem for Digraphs?

For an undirected graph G, a result known as Johansson's theorem improves on the upper bound on the chromatic number from Brooks's theorem when G is triangle-free.

Theorem (Johansson's theorem)

Suppose each vertex in G has degree at most d and G is triangle-free. Then $\chi(G) \sim O(d/\log(d))$.

Johansson's Theorem for Digraphs?

For an undirected graph G, a result known as Johansson's theorem improves on the upper bound on the chromatic number from Brooks's theorem when G is triangle-free.

Theorem (Johansson's theorem)

Suppose each vertex in G has degree at most d and G is triangle-free. Then $\chi(G) \sim O(d/\log(d))$.

It is currently unknown whether any analogue of this theorem holds for digraphs, *even in the classical setting*.

Johansson's Theorem for Digraphs?

For an undirected graph G, a result known as Johansson's theorem improves on the upper bound on the chromatic number from Brooks's theorem when G is triangle-free.

Theorem (Johansson's theorem)

Suppose each vertex in G has degree at most d and G is triangle-free. Then $\chi(G) \sim O(d/\log(d))$.

It is currently unknown whether any analogue of this theorem holds for digraphs, *even in the classical setting*.

Question: Directed Johansson's theorem?

[Erdős–Neumann-Lara, 1979] Suppose D is a digraph with no bidirected edges. Assume each vertex in D has maximum degree at most d. Then is $\vec{\chi}(D) \sim o(d)$?

Johansson's Theorem for Digraphs?

Question: Directed Johansson's theorem?

[Erdős–Neumann-Lara, 1979] Suppose D is a digraph with no bidirected edges. Assume each vertex in D has maximum degree at most d. Then is $\overrightarrow{\chi}(D) \sim o(d)$?

We already know that the answer to this question in the Borel context is no, even when the underlying undirected graph of D is triangle-free.

Johansson's Theorem for Digraphs?

Question: Directed Johansson's theorem?

[Erdős–Neumann-Lara, 1979] Suppose D is a digraph with no bidirected edges. Assume each vertex in D has maximum degree at most d. Then is $\overrightarrow{\chi}(D) \sim o(d)$?

We already know that the answer to this question in the Borel context is no, even when the underlying undirected graph of D is triangle-free.

It may be instructive to consider the μ-measurable or Baire-measurable setting.

Other Questions

1. What can we say about the descriptive combinatorics of hypergraphs?

Other Questions

1. What can we say about the descriptive combinatorics of hypergraphs?
2. What can we say about LOCAL algorithms for dicoloring? Are there implications between descriptive digraph combinatorics and LOCAL dicoloring algorithms?

Thank you!