ORBIT EQUIVALENCE & WREATH PRODUCTS

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Caltech Logic Seminar, May 17, 2021

DEF Let $G \curvearrowright (X,\mu)$ & $H \curvearrowright (Y,\nu)$ be free pmp actions of countable groups. The actions are called:

- Orbit equivalent (OE) if $\mathcal{R}(G \curvearrowright (X,\mu)) \cong \mathcal{R}(H \curvearrowright (Y,\nu))$
  i.e., there exists a measure space isomorphism $\phi : (X,\mu) \to (Y,\nu)$
  with $\phi(Gx) = H\phi(x)$ for a.e. $x \in X$.
- Stably orbit equivalent (SOE) if there exist positive measure sets $A \subseteq X$ & $B \subseteq Y$ s.t. $\mathcal{R}(G \curvearrowright (X,\mu)) \restriction A \cong \mathcal{R}(H \curvearrowright (Y,\nu)) \restriction B$

DEF The groups G & H are called OE if they admit free pmp actions which are OE.
They are called measure equivalent if they admit such action which are SOE

OPEN PROBLEM | Do there exist ICC groups G & H such that $LG \cong LH$ but G & H are not ME?

group vN algebra

Positive answer in the groupoid setting (Popa-Vaes):

Let G be a residually finite property (T) group whose FC-center is not virtually abelian (Ershov: G exists!)

profinite completion

Consider $\mathcal{G} = G \ltimes (\hat{G}, \hat{\mu})$. Then $L(\mathcal{G}) \cong L(\mathcal{G} \times S_{<\infty}) \cong L(\mathcal{G}) \otimes L(S_{<\infty})$

But $\mathcal{G}$ is not ME to $\mathcal{G} \times S_{<\infty}$ (since property (T) is an ME invariant & $G \times S_{<\infty}$ does not have (T))

THM (Bowen) Any two Bernoulli shifts, $F \curvearrowright (A^F, \mu^F)$ & $F \curvearrowright (B^F, \nu^F)$, over a free group F are OE.

COR $L(C_n \wr F) \cong L(F \ltimes (C_n^F, \mu_n^F)) \cong L(F \ltimes (C_m^F, \mu_m^F)) \cong L(C_m \wr F)$

Recall: $H \wr G = (\bigoplus_G H) \rtimes G$ is the wreath product of H by G

So $L(C_n \wr F) \cong L(C_m \wr F)$. **Q** Are $C_n \wr F$ & $C_m \wr F$ ME?

THM 1 (TD-W) Let A & B be two nontrivial finite groups & let F be a free group. Then the wreath product actions

$A \wr F \curvearrowright (A^F, \mu_A^F)$ & $B \wr F \curvearrowright (B^F, \mu_B^F)$ are OE.

Here, $A \wr F = (\bigoplus_F A) \rtimes F \curvearrowright \prod_F A$ as expected: $\bigoplus_F A$ acts by translation (viewed as a subgroup of $\prod_F A$) & F acts by shift.

- Bowen's THM is proved by showing that an OE between actions of a free factor of F can be lifted to obtain an OE of the corresponding coinduced actions of F

$$F = \underbrace{\langle a \rangle}_{\text{free factor of } F} * \langle b \rangle$$

$$\begin{tikzcd}
\langle a \rangle \curvearrowright (A^{\langle a \rangle}, \mu_A^{\langle a \rangle}) \arrow[d, "\text{coinduction}"] \arrow[r, phantom, "\underset{\text{(by Dye)}}{OE}"] & \langle a \rangle \curvearrowright (B^{\langle a \rangle}, \mu_B^{\langle a \rangle}) \arrow[d, "\text{coinduction}"] \\
F \curvearrowright (A^F, \mu_A^F) \arrow[r, phantom, "OE"] & F \curvearrowright (B^F, \mu_B^F)
\end{tikzcd}$$

This strategy cannot be applied to the wreath product case!

- $A \wr F \curvearrowright (A^F, \mu_A^F)$ is not coinduced from $A \wr \langle a \rangle \curvearrowright (A^{\langle a \rangle}, \mu_A^{\langle a \rangle})$
- $A \wr \langle a \rangle$ is not a free factor of $A \wr F$

Dye's Theorem with a cocycle on top

THM (Golodets-Sinelshchikov) Let R_0 & R_1 be aperiodic ergodic hyperfinite pmp equivalence relations on (X_1, μ_1) & (X_2, μ_2) respectively. For $i=0,1$ let $\alpha_i : R_i \to G$ be cocycles with dense range into a group G. Then there exists an OE $\phi : X_0 \to X_1$ from R_0 to R_1 such that the image of α_0 under ϕ is cohomologous to α_1.

COR (Feldman-Sutherland-Zimmer) Let G_0 & G_1 be infinite amenable groups and let $N_0 \trianglelefteq G_0$ & $N_1 \trianglelefteq G_1$ be normal subgroups. $\boxed{\text{Assume that } G_0/N_0 \cong_\pi G_1/N_1}$.
Let $G_0 \curvearrowright (X_0, \mu_0)$ be a free pmp action in which N_0 acts ergodically, & let $G_1 \curvearrowright (X_1, \mu_1)$ be a free pmp action in which N_1 acts ergodically.
Then there exists an OE ϕ of the actions such that $\phi(gN_0 x) = \pi(gN_0)\phi(x)$ a.e. In particular ϕ takes R_{N_0} to R_{N_1}

Applying this to $\underbrace{A \wr \mathbb{Z}}_{G_0} \curvearrowright (A^{\mathbb{Z}}, \mu_A^{\mathbb{Z}})$ & $\underbrace{B \wr \mathbb{Z}}_{G_1} \curvearrowright (B^{\mathbb{Z}}, \mu_B^{\mathbb{Z}})$

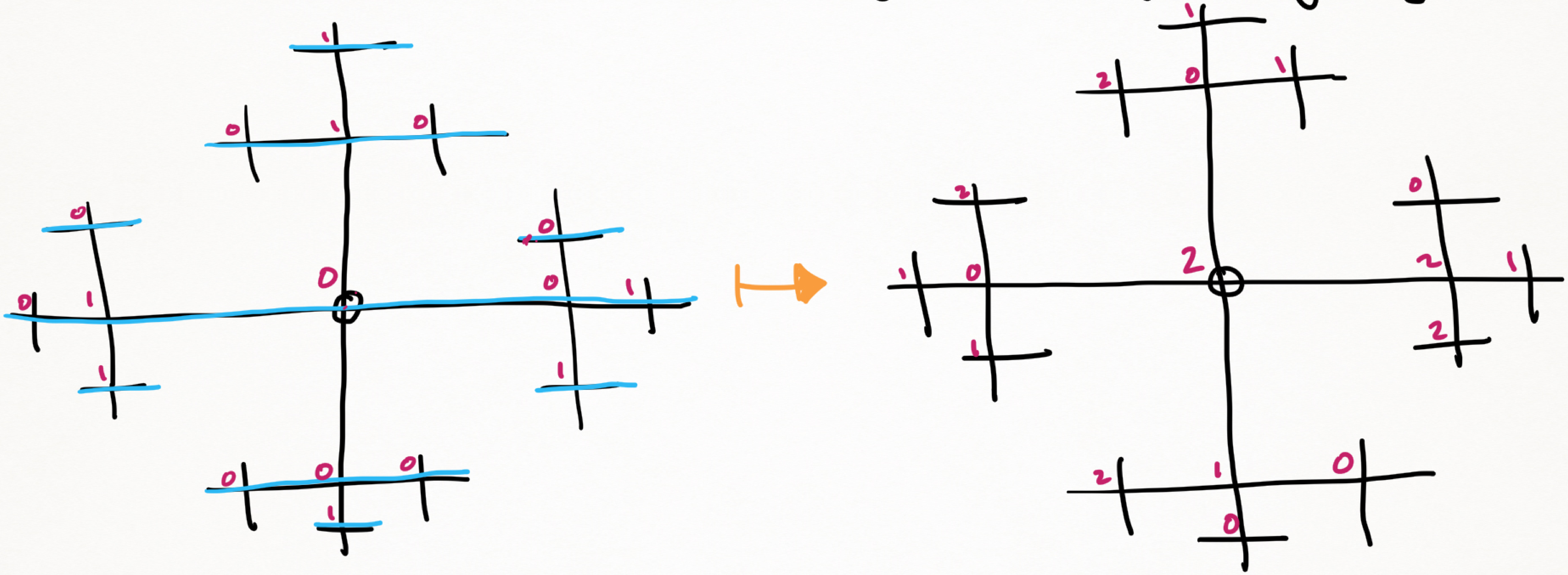
$N_0 = \oplus_{\mathbb{Z}} A$ $\quad$ $N_1 = \oplus_{\mathbb{Z}} B$

we obtain an OE $\phi : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ satisfying, for all $g \in \mathbb{Z}$:

if $x \sim y$ then $\phi(gx) \sim g\phi(y)$

where $\sim$ denotes equality on all but finitely many coordinates.

We apply this ϕ coset by coset (identifying $\mathbb{Z}$ with $\langle a\rangle$ in $F = \langle a\rangle * \langle b\rangle$)
to obtain a map $\Phi : A^F \to B^F$
that still satisfies $x \sim y \Rightarrow \Phi(gx) \sim g\Phi(y)$ ∎



Actually, we can view $A2F \curvearrowright (A^F, \mu_A^F)$ as being obtained from $A2\mathbb{Z} \curvearrowright (A^{\mathbb{Z}}, \mu_A^{\mathbb{Z}})$ by a generalized coinduction process.

In fact, the "coset by coset lifting process" is also used in Bowen's proof.

OPEN PROBLEM | Prove a simultaneous generalization of Bowen's THM & our THM 1.

THM 2 (TD-W) Let G be a sofic group with property (T) and no nontrivial finite normal subgroups (e.g. $SL_3(\mathbb{Z})$). Let A & B be finite groups with $|A| \neq |B|$. Then $A \wr G \curvearrowright (A^G, \mu_A^G)$ and $B \wr G \curvearrowright (B^G, \mu_B^G)$ are NOT OE (in fact, not even SOE).

- Use Popa cocycle superrigidity, sofic entropy, & some tricks to show that, for $|A| < |B|$, there is no embedding of $R(G \curvearrowright (A^G, \mu_A^G))$ into $R(B \wr H \curvearrowright (B^H, \mu_B^H))$ for any group H

ME and amenable extensions of free groups.

In general, what are the OE/ME-classes of groups of the form $K \rtimes F$, where K is infinite amenable, & F free?

Monod-Shalom If ϕ is an OE between free pmp actions $K_0 \rtimes F \curvearrowright (X_0, \mu_0)$ & $K_1 \rtimes F \curvearrowright (X_1, \mu_1)$ then, after composing with a full group element, ϕ takes R_{K_0} to R_{K_1}.

Using this we can show that $A \wr F$ is not ME with $\mathbb{Z} \times F$.

Conjecture For K infinite amenable:

$K \rtimes F$ is ME with $\mathbb{Z} \times F$

iff $\Rightarrow$ $K \rtimes F \curvearrowright K$ admits an invariant mean

Open Problem Suppose G is ME with $\mathbb{Z} \times F$.

Does G necessarily contain an infinite commensurated amenable subgroup?