Fraïssé classes with simply characterized big Ramsey degrees

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Subtitle:

Applying model theory and set theory to obtain results in structural Ramsey theory

Ramsey's Theorem

Ramsey's Theorem. Given any $k, \ell \geq 1$ and a coloring on the collection of all k-element subsets of ω into ℓ colors, there is an infinite $M \subseteq \omega$ such that each k-element subset of M has the same color.

Ramsey's theorem can be thought of as a structural Ramsey theorm coloring the k-hyperedges of a complete k-hypergraph on ω many vertices.

Does Ramsey's Theorem have analogues for other infinite structures?

$(\mathbb{Q}, <)$ has an approximate Infinite Ramsey Theorem

Thm. (Laver (upper bounds, unpub.), Devlin (exact bounds) 1979)
Given $k \geq 1$, there is a number $T(k, \mathbb{Q})$ such that for each coloring of the k-element subsets of $\mathbb{Q}$ into finitely many colors, there is a subcopy of $\mathbb{Q}$ with no more than $T(k, \mathbb{Q})$ colors.

$T(k, \mathbb{Q})$ is called the big Ramsey degree of k in $\mathbb{Q}$.

$T(k, \mathbb{Q})$ arises from an interplay of the ordering $<$ on $\mathbb{Q}$ and a well-ordering of $\mathbb{Q}$ in order-type ω. They are actually tangent numbers.

Let $q_0 < q_1 < q_2 < \dots$ well-order $\mathbb{Q}$.

Sierpiński's coloring of $[\mathbb{Q}]^2$:

$$c(\{q_m, q_n\}) = \begin{cases} \text{red} & \text{if } q_m <_{\mathbb{Q}} q_n \\ \text{blue} & \text{if } q_n <_{\mathbb{Q}} q_m \end{cases}$$

Big Ramsey Degrees on Infinite Structures

Def. [KPT 2005] Given an infinite structure $\mathbf{S}$ and a finite substructure $\mathbf{A} \leq \mathbf{S}$, we let $T(\mathbf{A}, \mathbf{S})$ denote the least T (if it exists) such that for any integer $\ell \geq 1$, given any coloring of $\binom{\mathbf{S}}{\mathbf{A}}$ into ℓ colors, there is a substructure $\mathbf{S}'$ of $\mathbf{S}$, isomorphic to $\mathbf{S}$, such that $\binom{\mathbf{S}'}{\mathbf{A}}$ takes no more than T colors.

When it exists, $T(\mathbf{A}, \mathbf{S})$ is called the big Ramsey degree of $\mathbf{A}$ in $\mathbf{S}$.

$\mathbf{S}$ has finite big Ramsey degrees if $T(\mathbf{A}, \mathbf{S})$ exists for each finite substructure $\mathbf{A} \leq \mathbf{S}$. The big Ramsey degrees in $\mathbf{S}$ are characterized if there is a means for computing them.

Motivation: Longstanding. More recent, [Zucker 2019].
("Big" relates to requiring an isomorphic copy of the infinite structure.)

Big Ramsey Degrees: Brief History

- Infinite complete k-hypergraph: All BRD = 1. (Ramsey 1929)
- $T(2,\mathbb{Q}) \geq 2$ (Sierpiński 1933). $T(2,\mathbb{Q}) = 2$ (Galvin unpub.)
- $T(\text{Edge, Rado}) \geq 2$ (Erdős, Hajnal, Pósa 1975)
- The rationals: BRD computed. (Devlin 1979)
- The K_3-free generic graph $\mathcal{H}_3$: $T(1,\mathcal{H}_3) = 1$. (Komjáth, Rödl 1986)
- The K_n-free generic graph $\mathcal{H}_n$: $T(1,\mathcal{H}_n) = 1$. (El-Zahar, Sauer 1989)
- $T(\text{Edge, Rado}) = 2$. (Pouzet, Sauer 1996)
- $T(\text{Edge}, \mathcal{H}_3) = 2$ (Sauer 1998)
- The Rado graph, etc.: BRD characterized. (Laflamme, Sauer, Vuksanović 2006). BRD computed. (J. Larson 2008)
- The countable ultrametric Urysohn space: BRD computed. (Nguyen Van Thé 2008)
- $\mathbb{Q}_n$ and the directed graphs $\mathbf{S(2)}$, $\mathbf{S(3)}$. BRD computed. (Laflamme, Nguyen Van Thé, Sauer 2010)

Big Ramsey Degrees: Recent Work

- The k-clique-free generic graph $\mathcal{H}_k$: Finite BRD (Dobrinen 2020 and 2019*)
  developed method of coding trees and related forcings
- 3-regular hypergraphs: Finite BRD (Balko, Chodounský, Hubička, Konečný, Vena 2019)
  used product Milliken theorem
- Universal structures, some metric spaces: Finite BRD (Mašulović 2020)
  used category theory
- $\mathbf{S}(n)$ for all $n \geq 2$: BRD calculated. (Barbosa 2020*)
  used category theory

Structures with finite big Ramsey degrees: Current Work

- Binary relational structures with free amalgamation, omitting finitely many irreducible substructures: Finite BRD (Zucker 2020*)
  used coding trees and forcing, and developed abstract approach

- Partial order, metric spaces, etc.: Finite BRD. (Hubička 2020*)
  used parameter words, first forcing-free proof for $\mathcal{H}_3$

- Fraïssé structures with SDAP$^+$: BRD characterized.
  (Coulson, Dobrinen, Patel 2020*)
  develop coding trees of 1-types, first envelope-free proof

- Binary relational free amalgamation classes with finitely many forbidden irreducible
  substructures: BRD characterized.
  (Balko, Chodounsky, Dobrinen, Hubička, Konečny, Vena, Zucker 2021*)
  various approaches

Other extensions in the works.

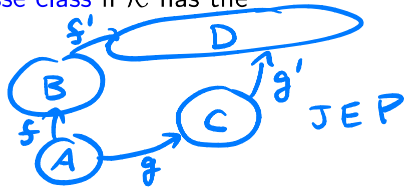
Fraïssé classes with simply characterized

big Ramsey degrees

Fraïssé Classes

A class $\mathcal{K}$ of finite structures is a Fraïssé class if $\mathcal{K}$ has the

- hereditary property
- joint embedding property
- amalgamation property

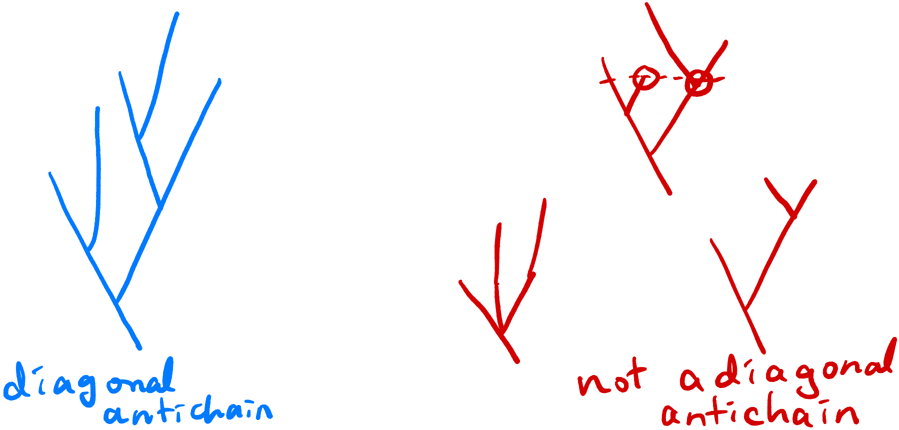


Given a Fraïssé class $\mathcal{K}$, there is a unique (up to isomorphism) countably infinite structure $\mathbf{K} := \mathrm{Flim}(\mathcal{K})$ which is ultrahomogeneous and universal for $\mathcal{K}$.

We will be working with Fraïssé classes with finitely many relations of any arity.

Diagonal Antichains in a finitely branching tree

An antichain A in a finitely branching tree is diagonal if the meet closure $A^{\wedge}$ of A has splitting nodes with degree 2, and the lengths among the splitting and terminal nodes are all distinct. (draw)

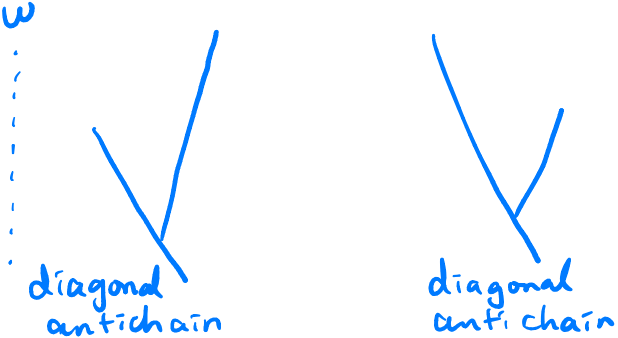


Prototype example of simply characterized BRD

The lex order on $2^{<\omega}$ represents $(\mathbb{Q},<)$

$\mathcal{LO}$ = Fraïssé class of finite linear orders. Fraïssé limit is $(\mathbb{Q},<)$.

Devlin types. (draw)

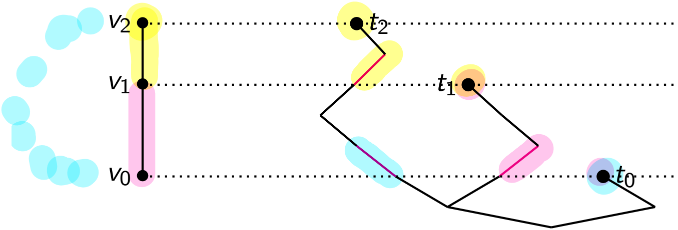


These represent the 2 colors of Sierpiński's colorin

Second prototype example: Finite graphs

The Rado graph is the Fraïssé limit of the class of finite graphs.

Given a graph **A** with vertices $\langle v_n : n < N \rangle$, a set of nodes $\{t_n : n < N\} \subseteq 2^{<\omega}$ codes **A** if and only if for each $m < n < N$, $v_n \; E \; v_m \Leftrightarrow t_n(|t_m|) = 1$. $t_n(|t_m|)$ is called the passing number of t_n at t_m.



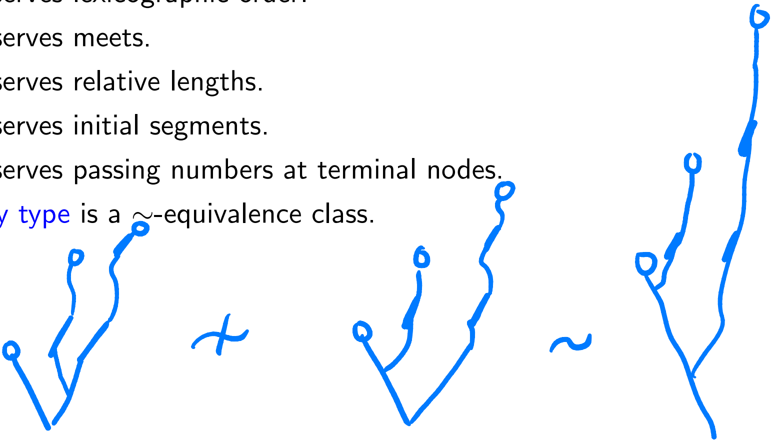
Similarity types for graphs (LSV)

Let A and B be antichains in $2^{<\omega}$. We say that A and B are similar ($A \sim B$) iff there is a bijection $f : A^\wedge \to B^\wedge$ satisfying the following:

1. f preserves lexicographic order.
2. f preserves meets.
3. f preserves relative lengths.
4. f preserves initial segments.
5. f preserves passing numbers at terminal nodes.

A similarity type is a $\sim$-equivalence class.

codes a triangle



Graphs have simply characterized big Ramsey degrees

Let $\mathcal{G}$ be the class of finite graphs, and let $\mathbf{R}$ denote its Fraïssé limit, the Rado graph.

Theorem. (Sauer 2006) and (Laflamme, Sauer, Vuksanovic 2006) The Fraïssé class of finite graphs has finite big Ramsey degrees in the Rado graph. Moreover, given $\mathbf{A} \in \mathcal{G}$, $T(\mathbf{A}, \mathbf{R})$ is the number of similarity types of diagonal antichains representing a copy of $\mathbf{A}$.

Proof uses $2^{<\omega}$ to code a universal graph, Milliken's theorem on strong tree envelopes of diagonal antichains, and a diagonal antichain in $2^{<\omega}$ coding the Rado graph.

Extensions in LSV

[Laflamme, Sauer, Vuksanovic 2006] further characterize the big Ramsey degrees for the random digraph, the random tournament, and more generally, structures with finitely many binary relations and a "universal constraint set" consisting of structures on universes of size two.

$k = 2^n$ where n is the number of binary relations

Proof uses trees $k^{<\omega}$, for some finite k, which codes a universal structure into which the Fraïssé limit embeds, envelopes, Milliken's theorem, and a diagonal antichain coding the Fraïssé structure.

(CDP) extends (LSV) to certain classes of relational structures of any arity, but with a more streamlined proof.

Fraïssé classes with simply characterized BRD

Work by Coulson, D., and Patel was motivated by the following questions:

(Sauer, BIRS 2018) Can the forcing (done by D. for Henson graphs) be done directly on the Fraïssé structures?

(D.) For which Fraïssé classes can we prove finite big Ramsey degrees with reasonably simple forcing arguments (as compared with the Henson graphs)?

Coding trees of 1-types

In (CDP), we move from trees on $k^{<\omega}$ to trees of 1-types over initial structures of a fixed enumerated Fraïssé structure.

Let $\mathbf{K}$ be the Fraïssé limit of a given Fraïssé class $\mathcal{K}$ with enumerated vertices $\langle v_n : n < \omega \rangle$. Let $\mathbf{K}_n$ denote $\mathbf{K} \restriction \{v_i : i < n\}$.

The coding tree of 1-types $\mathbb{S}(\mathbf{K})$ is the set of all complete quantifier-free 1-types over initial segments of $\mathbf{K}$ along with a function $c : \omega \to \mathbb{S}(\mathbf{K})$ such that $c(n)$ is the 1-type of v_n over $\mathbf{K}_n$. The tree-ordering is simply inclusion.

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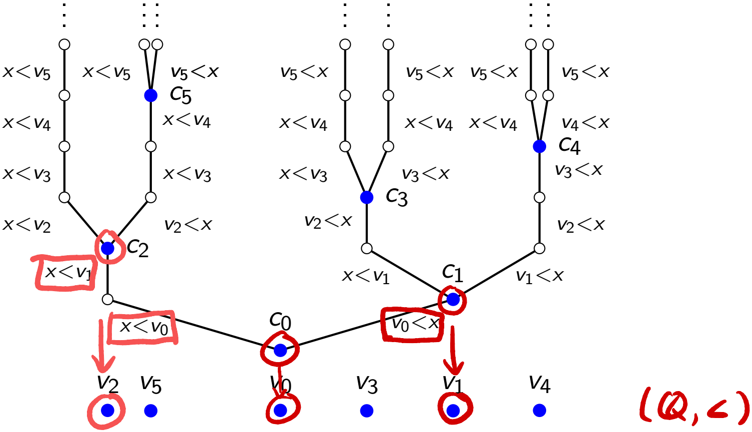
Members of $\mathbb{S}(\mathbf{K})$ can be thought of as sequences of partial 1-types:

$s(0)$ is the 1-type over the empty structure such that $s(0) \subseteq s$.

For $1 \leq i \leq n$, $s(i)$ is the set of formulas in $s \restriction \mathbf{K}_i$ that are not in $s \restriction \mathbf{K}_{i-1}$.

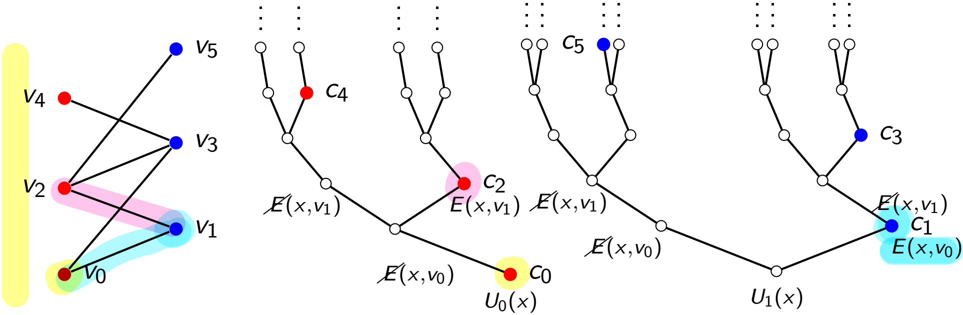
Each $s \in \mathbb{S}(\mathbf{K})$ determines a unique sequence $\langle s(i) : i < |s| \rangle$, where $\{s(i) : i < |s|\}$ forms a partition of s.

Coding Tree of 1-types for $(\mathbb{Q}, <)$



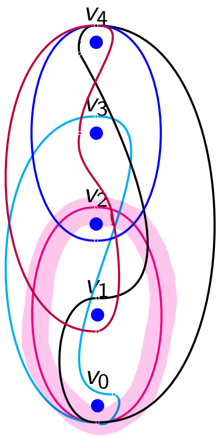
$c_0 = \emptyset$. $c_1 = \{(v_0 < x)\}$. $c_2 = \{(x < v_0), (x < v_1)\}$.
$c_3 = \{(v_0 < x), (x < v_1), (v_2 < x)\}$.

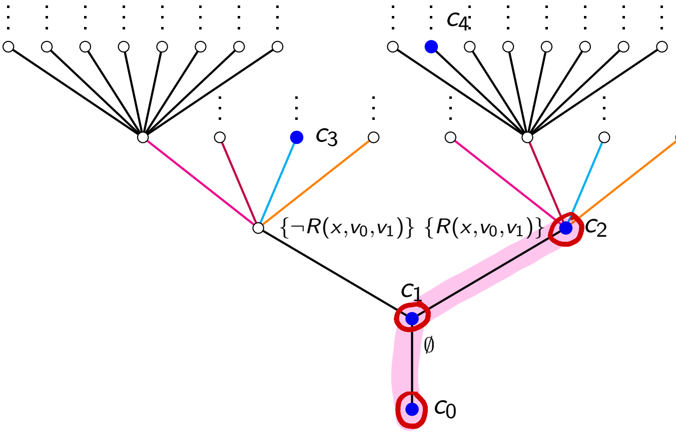
Coding tree of 1-types for the generic bipartite graph



$c_0 = \{U_0(x)\}$. $c_1 = \{U_1(x), E(x, v_0)\}$.
$c_2 = \{U_0(x)\}, \neg E(x, v_0), E(x, v_1)\}$.
$c_3 = \{U_1(x), E(x, v_0), \neg E(x, v_1), E(x, v_2)\}$.

Coding tree of 1-types for the generic 3-regular hypergraph





$c_2 = \{R(x, v_0, v_1)\}$. $c_3 = \{\neg R(x, v_0, v_1), R(x, v_0, v_2), \neg R(x, v_1, v_2)\}$.

Note: Every substructure $\mathbf{A}$ of $\mathbf{K}$ is coded by the coding nodes in $\mathbb{S}(\mathbf{K})$ representing the vertices of $\mathbf{A}$.

Passing Types

Given $s, t \in \mathbb{S}$ with $|s| < |t|$, $t(|s|)$ is the set of all formulas in $t \restriction \mathbf{K}_{|s|}$,
$t(|s|)$ is the passing type of t at s.

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Given $A \subseteq \mathbb{S}$, $t, c_n \in \mathbb{S}$ with $|c_n| < |t|$, $t(c_n; A)$ denotes the passing type of t at c_n over A: those formulas in $t(|c_n|)$ in which all parameters are from among vertices represented by coding nodes in A with length less than $|c_n|$, along with v_n.

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Given $A, B \subseteq \mathbb{S}$ and m, n, write $s(c_m; A) \sim t(c_n; B)$ whenever there is a bijection between the coding nodes in A of length less than $|c_m|$ and the coding nodes in B of length less than $|c_n|$, and the order-preserving bijection between those vertices takes $s(c_m; A)$ to $t(c_n; B)$.

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Fact. If A and B have coding nodes $\langle c_n^A : n < N \rangle$ and $\langle c_n^B : n < N \rangle$, and for all $m < n < N$, $c_n^A(c_m^A; A) \sim c_n^B(c_m^B; B)$, then the subtructures of $\mathbf{K}$ represented by A and B are order-isomorphic.

Similarity Maps in Coding Trees of 1-types

Let S and T be meet-closed subsets of $\mathbb{S}$. A function $f : S \to T$ is a similarity map from S to T if for all nodes $s, t \in \mathbb{S}$, the following hold:

1. f is a bijection which preserves the lexicographic order in $\mathbb{S}$.
2. f preserves meets, and hence splitting nodes.
3. f preserves relative lengths.
4. f preserves initial segments.
5. f preserves coding nodes and their parameter-free formulas.
6. f preserves relative passing types at coding nodes: $s(c^S_n; S) \sim f(s)(c^T_n; T)$, for each n such that $|c^S_n| < |s|$.

We write $S \sim T$ whenever there exists a similarity map from S to T. A similarity type is a $\sim$-equivalence class.

Simply characterized big Ramsey degrees

We say that a Fraïssé class $\mathcal{K}$ has simply characterized big Ramsey degrees if,

letting $\mathbf{K}$ be an enumerated Fraïssé limit of $\mathcal{K}$ and $\mathbb{S}(\mathbf{K})$ be the coding tree of (quantifier-free) 1-types,

for each $\mathbf{A} \in \mathcal{K}$, $T(\mathbf{A}, \mathbf{K})$ is exactly the number of similarity types of diagonal antichains of coding nodes in $\mathbb{S}(\mathbf{K})$ representing $\mathbf{A}$.

We now look at an amalgamation property which ensures simply characterized big Ramsey degrees.

Substructure Free Amalgamation Property

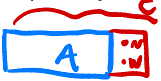
A Fraïssé class $\mathcal{K}$ satisfies SFAP if $\mathcal{K}$ has free amalgamation, and given $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{K}$, the following holds: Suppose

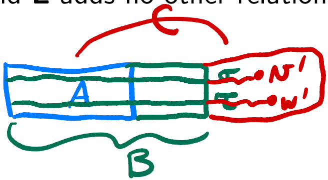
(1) $\mathbf{A}$ is a substructure of $\mathbf{C}$, where $\mathbf{C}$ extends $\mathbf{A}$ by two vertices, say $\mathrm{C} \setminus \mathrm{A} = \{v, w\}$;

(2) $\mathbf{A}$ is a substructure of $\mathbf{B}$ and σ and τ are 1-types over $\mathbf{B}$ with $\sigma \upharpoonright \mathbf{A} = \mathrm{tp}(v/\mathbf{A})$ and $\tau \upharpoonright \mathbf{A} = \mathrm{tp}(w/\mathbf{A})$; and

(3) $\mathbf{B}$ is a substructure of $\mathbf{D}$ which extends $\mathbf{B}$ by one vertex, say v', such that $\mathrm{tp}(v'/\mathbf{B}) = \sigma$.

Then there is an $\mathbf{E} \in \mathcal{K}$ extending $\mathbf{D}$ by one vertex, say w', such that $\mathrm{tp}(w'/\mathbf{B}) = \tau$, $\mathbf{E} \upharpoonright (\mathrm{A} \cup \{v', w'\}) \cong \mathbf{C}$, and $\mathbf{E}$ adds no other relations over D.





SFAP = Free 3-Amalgamation

A generalization of SFAP to Disjoint Amalgamation classes

$\mathcal{K}$ satisfies SDAP if $\mathcal{K}$ has disjoint amalgamation, and given $\mathbf{A}, \mathbf{C} \in \mathcal{K}$ with $\mathbf{A}$ a substructure of $\mathbf{C}$, where $\mathrm{C} \setminus \mathrm{A} = \{v, w\}$, there exist $\mathbf{A}', \mathbf{C}' \in \mathcal{K}$, with $\mathbf{A}$ a substructure of $\mathbf{A}'$ and $\mathbf{C}'$ a disjoint amalgamation of $\mathbf{A}'$ and $\mathbf{C}$ over $\mathbf{A}$, such that letting $\{v', w'\} = \mathrm{C}' \setminus \mathrm{A}'$ and assuming

(1) $\mathbf{B} \in \mathcal{K}$ is any structure containing $\mathbf{A}'$ as a substructure, and let σ and τ be 1-types over $\mathbf{B}$ satisfying $\sigma \restriction \mathbf{A}' = \mathrm{tp}(v'/\mathbf{A}')$ and $\tau \restriction \mathbf{A}' = \mathrm{tp}(w'/\mathbf{A}')$,

(2) $\mathbf{D} \in \mathcal{K}$ extends $\mathbf{B}$ by one vertex, say v'', such that $\mathrm{tp}(v''/\mathbf{B}) = \sigma$,

Then there is an $\mathbf{E} \in \mathcal{K}$ extending $\mathbf{D}$ by one vertex, say w'', such that $\mathrm{tp}(w''/\mathbf{B}) = \tau$ and $\mathbf{E} \restriction (\mathrm{A} \cup \{v'', w''\}) \cong \mathbf{C}$.

SDAP$^+$: SDAP + $\exists$ diagonal coding tree + Extension Property

Prop. SFAP implies SDAP$^+$.

Examples

The following Fraïssé classes satisfy SFAP:

- graphs, ordered graphs, graphs with finitely many edge relations, etc.
- *n*-partite graphs
- all free amalgamation classes with finitely many relations in which any forbidden substructures are 3-irreducible.
  (this includes k-hypergraphs for any $k \geq 3$)

The following Fraïssé classes have limits satisfying SDAP^+:

- linear orders (and main reducts)
- Fraïssé classes with SFAP with an additional linear order
- convexly ordered equivalence relations, and coarsening hierarchy
- extensions of LSV to higher arities

SDAP$^+$ implies simply characterized big Ramsey degrees

Theorem. (CDP) Suppose $\mathcal{K}$ is a Fraïssé relational class with finitely many relations satisfying SFAP, or just SDAP$^+$. Given $\mathbf{A} \in \mathcal{K}$, the big Ramsey degree of $\mathbf{A}$, $T(\mathbf{A}, \mathbf{K})$, equals the number of similarity types diagonal antichains of coding nodes in $\mathbb{S}(\mathbf{K})$ representing a copy of $\mathbf{A}$.

SDAP$^+$ $\Longrightarrow$ simply characterized BRD: Proof Ideas

I. If $\mathcal{K}$ satisfies SFAP, or just SDAP$^+$, then there is a diagonal subtree $\mathbb{T}$ of $\mathbb{S}(\mathbf{K})$ which again codes a copy of $\mathbf{K}$.

II. Do forcing arguments over $\mathbb{T}$. Given $\mathbf{A} \in \mathcal{K}$, show that $T(\mathbf{A}, \mathbf{K})$ is bounded above by the number of similarity types of diagonal antichains coding $\mathbf{A}$. Note: The forcing just conducts an unbounded search for a finite object - we never pass to a generic extension.

III. Take an antichain $\mathbb{D}$ of coding nodes in $\mathbb{T}$ which represents a copy of $\mathbf{K}$. Prove that for each $\mathbf{A} \in \mathcal{K}$, all similarity types of antichains in $\mathbb{D}$ coding $\mathbf{A}$ persist in any subset of $\mathbb{D}$ which again codes a copy of $\mathbf{K}$.

Fun Fact: This approach bypasses any need for envelopes.

SDAP$^+$ $\Longrightarrow$ simply characterized BRD: Proof Ideas

I. If $\mathcal{K}$ satisfies SFAP, or just SDAP$^+$, then there is a diagonal subtree $\mathbb{T}$ of $\mathbb{S}(\mathbf{K})$ which again codes a copy of $\mathbf{K}$.

$\text{SDAP}^+ \implies$ simply characterized BRD: Proof Ideas

II. Do forcing arguments over $\mathbb{T}$ to prove a pigeonhole principle for level set extensions. Then by induction, show that for each $\mathbf{A} \in \mathcal{K}$, $T(\mathbf{A}, \mathbf{K})$ is bounded above by the number of similarity types of diagonal antichains coding $\mathbf{A}$.

SDAP$^+$ $\Longrightarrow$ simply characterized BRD: Proof Ideas

III. Take an antichain $\mathbb{D}$ of coding nodes in $\mathbb{T}$ representing a subcopy of **K**. $\mathbb{D} \subseteq \mathbb{T}$ implies $\mathbb{D}$ is diagonal. Prove that for each $\mathbf{A} \in \mathcal{K}$, all similarity types of antichains in $\mathbb{D}$ coding **A** persist in any subset of $\mathbb{D}$ which again codes a copy of **K**. This gives the first proof of exact big Ramsey degrees without resorting to "envelopes".

$\mathrm{SDAP}^+ \implies$ simply characterized BRD: Proof Ideas

Fraïssé classes with simply characterized BRD

Theorem. (CDP) The following Fraïssé limits have simply characterized big Ramsey degrees:

(1) Forb($\mathcal{F}$), where $\mathcal{F}$ is a finite set of finite 3-irreducible structures in a finite relational language, without or with an additional linear order.
(2) Unrestricted Fraïssé classes (generalizing those in (LSV) to higher arities).
(3) n-partite graphs, for any $n \geq 2$.
(4) $\mathbb{Q}$, $\mathbb{Q}_n$, $\mathbb{Q}_{\mathbb{Q}}$, $(\mathbb{Q}_{\mathbb{Q}})_n$, the main reducts of $\mathbb{Q}$.

Simply Characterized Big Ramsey Degrees

Given a Fraïssé class $\mathcal{K}$ in a finite relational language $\mathcal{L}$ such that the Fraïssé limit $\mathbf{K}$ of $\mathcal{K}$ satisfies SDAP$^+$, given $\mathbf{A} \in \mathcal{K}$, $(\mathbf{A}, <)$ denote $\mathbf{A}$ together with a fixed enumeration $\langle \mathrm{a}_i : i < n \rangle$ of the universe of $\mathbf{A}$. T is a diagonal tree coding $(\mathbf{A}, <)$ iff

1. T is a tree with $2n - 1$ many levels, n terminal nodes, and branching degree two.
2. Let $\langle t_i : i < n \rangle$ enumerate the terminal nodes in T in order of increasing length. Given $i < j < k < n$, if t_j and t_k both extend the same node in T at the level of t_i, then τ_j and τ_k have the same 1-types over $\mathbf{A}_m$. That is, $\tau_j \restriction \mathbf{A}_m = \tau_k \restriction \mathbf{A}_m$.

Let $\mathcal{D}(\mathbf{A}, <)$ be the number of diagonal trees coding $(\mathbf{A}, <)$ and $\mathcal{OA}$ consist of one representative from each isomorphism class of ordered copies of $\mathbf{A}$. Then

$$T(\mathbf{A}, \mathbf{K}) = \sum_{(\mathbf{A}, <) \in \mathcal{OA}} \mathcal{D}(\mathbf{A}, <)$$

Galvin-Prikry Analogues

Working with antichains D coding subcopies of $\mathbf{K}$ inside diagonal coding trees, we can obtain analogues of the Galvin-Prikry Theorem.

Theorem. (D) Given a Fraïssé class $\mathcal{K}$ with SFAP, there is a Baire space $\mathcal{B}(\mathbf{K}) \subseteq [\omega]^\omega$ such that each point in $\mathcal{B}(\mathbf{K})$ represents a copy of $\mathbf{K}$, and all Borel subsets of $\mathcal{B}(\mathbf{K})$ are completely Ramsey.

In progress: One can moreover set up the spaces $\mathcal{B}(\mathbf{K})$ to recover the exact big Ramsey degrees for members of $\mathcal{K}$.
(answering a question of Todorcevic at Luminy 2019)

Ramsey Theory and Topological Dynamics

(Kechris, Pestov, Todorcevic 2005) The KPT Correspondence:
A Fraïssé class $\mathcal{K}$ has the Ramsey property iff $\mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$ is extremely amenable.

(Zucker 2019): If $\mathbf{K}$ admits a big Ramsey structure, then $\mathrm{Aut}(\mathbf{K})$ has a metrizable universal completion flow, which is unique up to isomorphism.

(CDP): Any Fraïssé structure satisfying SDAP^{+} admits a big Ramsey structure which is simply characterized by the addition of two more binary relations.

Fraïssé classes with not-as-simply characterized big Ramsey degrees

Binary relations and forbidden irreducible substructures

A structure **F** is irreducible if any two vertices in **F** are in some **F**-relation.

In (D. 20 and 19*), upper bounds were found for the k-clique-free Henson graphs.

In (Zucker 20*), upper bounds were found for Fraïssé limits of free amalgamation classes with finitely many binary relations and finitely many forbidden irreducible substructures.

Theorem. (BCDHKVZ* 21) Let $\mathcal{L}$ be a finite binary relational language. Let $\mathcal{F}$ be a set of finitely many finite irreducible $\mathcal{L}$-structures, and let Forb($\mathcal{F}$) be the Fraïssé class of finite $\mathcal{L}$ structures **A** such that no member of $\mathcal{F}$ embeds into **A**. Then the big Ramsey degrees of Forb($\mathcal{F}$) are characterized.

Example: Coding tree for $\mathcal{H}_3$

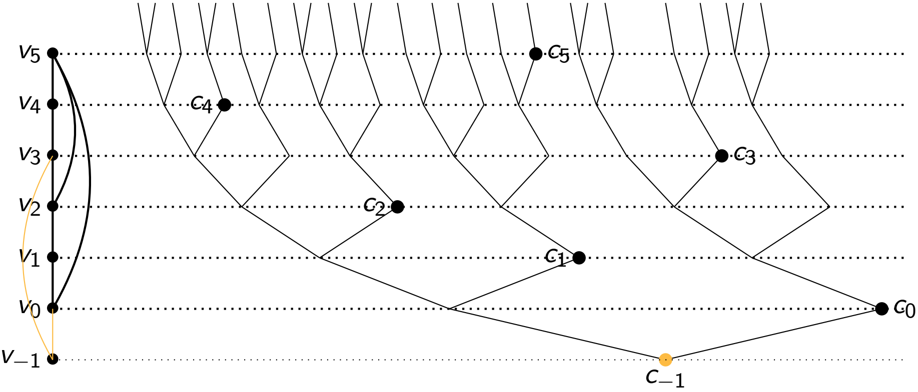


Figure: A coding tree of 1-types $\mathbb{S}_3$ coding $\mathcal{H}_3$

Exact BRD for triangle-free graphs

Theorem. (BCDHKVZ* 21) Let $\mathbf{A} \in \mathcal{G}_3$ be given. The big Ramsey degree of $\mathbf{A}$ in $\mathcal{H}_3$ is the number of ep-similiarity types of diagonal antichains of coding nodes in $\mathbb{S}_3$ representing a copy of $\mathbf{A}$.

ep-similarity is similarity plus keeping track of age changes due to two nodes coding an edge with a common vertex in $\mathbf{K}$.

Current Methods for BRD, and Future Directions

Current Methods:

- Milliken's Theorem and variations, (no forcing): (Devlin), (Laflamme, Sauer, Vuksanović), (Laflamme,Nguyen Van Thé, Sauer), (BCHKV).
- Coding trees (using forcing on diagonal subtrees, direct - no envelopes): (D), (CDP).
- Category theory: (Barbosa), (Mašulović).
- Coding trees (using forcing on Milliken-style coding trees): (Zucker), (BCDHKVZ).
- Parameter words: (Hubička)

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- Parameter words: (Hubička)

Future directions:

- BRD for free amalgamation classes more generally.
- BRD for strong amalgamation classes more generally.
- Infinite dimensional Ramsey theory: Rado graph done in (D. 2019*), Structures with $SDAP^+$ - in preparation (D. 2021*). Others?

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Thank you for your kind attention!